Integration by Parts Trick with the Gamma Function

Normally with the Gamma Function, you can use the trick of the numerator integrating to the denominator to solve most problems.

$\int x^2 e^{-x/2} dx$

As long as the boundaries goes from 0 to infinity, you can use this trick. However, if the boundaries were 3 to infinity, this trick would no longer work and you would need to do two iterations of parts.

You can use a slight trick to get around doing parts. Consider the following integral:

$\int x^{a-1} e^{-x/b} dx$

If you were to keep breaking this out into parts, it would be as followed:

$-e^{-x/b}(bx^{a-1}+b^2(a-1)x^{a-2}+b^3(a-1)(a-2)x^{a-3}+b^4(a-1)(a-2)(a-3)x^{a-4}\ldots b^{a-1}(a-1)(a-2)(a-3)\ldots(2)x^1+b^a(a-1)(a-2)(a-3)\ldots(2)(1))$

This can be written as a summation as n goes from 1 to a.

$-e^{-x/b}\sum b^n \frac{(a-1)!}{(a-n)} x^{a-n}$

However, you can also simply notice the pattern that occurs when you integrate a gamma as well. If you look at the parts integration for a gamma, you would get something like this.

$-u\int dv - du\int\int dv - d^2u\int\int\int dv - d^3u\int\int\int\int dv - d^4u\int\int\int\int\int dv - \ldots$

This is easy to see with an example.

$\int x^2 e^{-x} dx$

dv will always just yield a -1 factor no matter how many times you integrate it. You also will keep taking the derivative of u, x^2.

$-e^{-x/2}(x^2 + 2x + 2)$

Take the integral from the beginning now. In parts, dv would be $e^{-x/2}$, which will always yield an addional -2 when you integrate it. At the same time, you would start with u, x^2, and keep taking the derivative of this to get the other terms. See below

$-e^{-x/2}\{(2)x^2 + (2)(2)(2)x + (2)(2)(2)(2)\}$

Now let's see it with a problem. Find the probability that $x>3$ for the following gamma function

$\frac{1}{87480} x^5 e^{-x/3}$

This would be the integral from 3 to infinity of this pdf, which would require 5 uses of parts. However, using the trick described above, it can be broken down into the following form.

$\frac{1}{87480}[-e^{-x/3}(3x^5+(3)(3)(5)x^4+(3)(3)(3)(5)(4)x^3+(3)(3)(3)(3)(5)(4)(3)x^2+(3)(3)(3)(3)(3)(5)(4)(3)(2)x^1(3)(3)(3)(3)(3)(3)(5$

from 3 to infinity, which would equal $\frac{1}{87480}(0 - (-e^{-1}(237654)))$ which = .9994 probability.

Check this in wolfram alpha to see that it does indeed match,

$$\int_3^\infty \frac{x^5 e^{-\frac{x}{3}}}{87480}\,dx = \frac{163}{60e} \approx 0.999406$$

This is useful if you ever need to use parts for a gamma or exponential that does not go from 0 to infinity.

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