Economic and Policy Implications of Restricted Patch Distribution

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Abstract

In this paper, we study how restricting the availability of patches to legal users impacts vendor's profits, market share, software maintenance decisions, and welfare outcomes. Prior work on this topic assumes that hacker's effort is independent of the vendor's decision to release the patch freely or not. Clearly, if the patch is not available to everyone, the hacker finds it easier to exploit the vulnerability in the product and, as a result, is likely to alter his effort. In order to understand the role of a strategic hacker, we build a game-theoretic model, where the hacker's decision is endogenous. With this model, we find that the hacker's effort may, on one hand, decrease the utility that the vendor can extract from the consumers. On the other hand, it may help differentiate the legal version of the product from the pirated version. A vendor can strategically exploit the hacker's behavior in its pricing and software maintenance decisions. The endogeneity of hacker's actions drives several of our findings that have interesting policy implications. For example, the vendor may increase the price and reduce market share in order to exploit the differentiation. In such a case, there may be more pirates in the restricted-patch case than when the patch is freely available, a result that runs counter to typical arguments provided for restricting patches. A government body that understands this trade-off may exert a different level of piracy prevention effort so that the vendor is incentivized to make decisions that improve social welfare.

Key words : information security; patch distribution; countervailing incentive; public policy

1 Introduction

Software vendors have restricted the availability of patches only to *legal users* because of piracy concerns. Windows Genuine Advantage from Microsoft is an example of such a program. Patch restrictions also have implications on hackers' efforts as they seek to exploit vulnerabilities that the patches are designed to resolve. By restricting patch distribution, the vendor can indirectly control the hacker's action, which affects consumers' expected utility and their buying decisions. Thus, in addition to using the software price, vendor may use patch restriction to influence consumer behavior. The impact of a strategic hacker on the resulting trade-offs has not been explored before. Our aim is to derive managerial and policy insights by comparing two settings: one where patch access is provided only to legal users and the other where patches are distributed freely (to both legal users and pirates).

Patches can be categorized into two types (Lahiri, 2012): (a) *Security updates* which address vulnerabilities that third party agents (e.g., hackers) exploit to compromise the software, and (b) *Performance updates* which improve user experience (e.g., prevent system crashes). Because of our focus on strategic interactions between the vendor and the hacker, we primarily study security updates, although we briefly consider the performance updates later in the paper as well.

In order to limit piracy, government typically exerts effort by instituting laws, prosecuting offenders, etc. So, a vendor's decision to restrict updates is aimed at thwarting piracy above and beyond the governmental effort. The hacker's role in the market can be imagined as destroying/decreasing the consumer's willingness to pay. So, at first glance, one would imagine that as long as the marginal cost of effort in marginalizing the hacker is compensated by the corresponding increase in the price of the software, the vendor will make the effort. However, that is not necessarily the case. Our analysis finds that the hacker's action has two different effects on the vendor profit. The adverse effect–which is straightforward–decreases the consumer utility that the vendor can extract and, therefore, has negative implications on the vendor profit. It occurs independent of whether patches are restricted or not. The countervailing effect–which occurs only when the patches are restricted–is one where the hacker's actions are beneficial to the vendor profit. When the latter effect is dominant, the hacker's effort helps the vendor in further differentiating the legal and the pirated versions of the software. So, it creates an incentive for the vendor to not marginalize hacking activity, thereby improving the differentiation and, ultimately, his profit. This seemingly counterintuitive behavior of the vendor has some interesting welfare implications as well.

The main argument that vendors provide in favor of restricting patch distribution is that it encourages more of pirates to buy the software. However, we show that the vendor may increase the price of the software while restricting patch distribution, in effect lowering the market share and, in fact, increasing the incentive to pirate. Doing so, the vendor exploits the countervailing effect of hacker actions that was alluded to earlier. This vendor behavior does not necessarily reduce social welfare. Consider, for example, a situation where government exerts little antipiracy effort. Then, a market may not be sustainable if distribution of patches is not restricted. We show that the countervailing effect may provide enough differentiation via restricted patch distribution that allows a vendor to serve the market and generate social welfare. Consequently, the hacker may alleviate social planner's investment in antipiracy measures. We also find that there may be a disagreement between the social planner and the vendor regarding the policy of patch distribution when the social planner is willing to put in moderate amount of effort in antipiracy effort levels. The agreement is restored when the social planner is willing to exert significant antipiracy effort.

We deal with a parsimonious model involving two stages of strategic interactions. The vendor makes the pricing and the software maintenance decisions in the first stage, and the hacker and the users arrive at their decisions simultaneously in the second stage. Even for this stylized model, where hacker actions are endogenized, a closed-form analytical solution to the vendor's problem seems intractable. Nevertheless, we develop various analytical insights using comparative statics and numerical analyses.

This paper is organized as follows. In §2, we review the extant literature most relevant to this topic. Following that, in §3, we describe our model. In §4, we present our equilibrium analysis and §5 compares the two settings – one with no patch restriction and the other where patch distribution is restricted. Finally, we present our concluding remarks in §6.

2 Literature Review

Our research relates to software maintenance, information security, and piracy literatures. We provide a brief overview of each of these areas next. Towards the end of this section, we review two specific papers–August and Tunca (2008) and Lahiri (2012)–in the literature that are most closely related to the topic of our study.

In our model, the patch quality may be viewed as an outcome of the vendor's software maintenance resource decision. We recognize that several papers have focused solely on the resource allocation problem from various different perspectives. For example, Kulkarni et al. (2009) develops a queuing model for optimal resource allocations. Similarly, Ji et al. (2005) consider the joint determination of release time and the division of effort between constructing and debugging software so that the total cost is minimized. Jiang and Sarkar (2003) analyzes the optimal software release time where patching is explicitly considered and shows how the total cost can be reduced. To the best of our knowledge, ours is the first paper to consider the resource decision along with patch distribution policy restriction and its impact on piracy.

Our paper builds on extant literature on the economics of information security. Prior work has analyzed various facets of this domain. Arora et al. (2004) shows that vulnerability disclosures expedite the response from large vendors and subsequently benefit software users. Kannan and Telang (2005) shows that a market-based mechanism for vulnerability disclosure makes the market worse off than one without such a market. Arora et al. (2006) shows that a software vendor is better-off releasing a buggier software early and patching it later particularly when the market for the product is large. August and Tunca (2006) considers users' incentive to patch security flaws. They find that subsidy based patching policy performs better than mandatory or tax based patching policy. Png and Wang (2009a) considers the strategic interaction between end-users in taking security precautions and the interaction between end-users' and hackers' actions. Our paper borrows several key modeling details from Kannan and Telang (2005). Different from the prior information security literature, we investigate the possibility that the vendor uses the hacker's effort to improve his profit. We are not aware of any prior work that has focused on this aspect.

Our paper studies the patch restriction problem in the context of piracy. Most prior works on software piracy analyze the impact of piracy on the legitimate producer's sales and profit. A stream of prior papers argue that a producer may not have the incentive to eliminate piracy from the market (e.g., Chen and Png, 2003; Gopal and Sanders, 1997; Shy, 2001). For example, piracy generates network externality benefits which may lead to increased demand for the legitimate version (Conner and Rumelt, 1991; Shy, 2001). Pirated versions may also serve as a coordination device to reduce price competition (Jain, 2008). Sharing may be profitable for the producer if the transaction cost of sharing is lower than the marginal cost of production (Varian, 2000). The impact of piracy on social welfare has been shown to cut both ways. Strict rules to combat piracy have been shown to increase the producer's profit while reducing the benefits of utilizing already developed products (Chen and Png, 2003). Chen and Png (2003) contend that from the social welfare perspective, it is better to manage piracy through price cuts than strict enforcement. Gopal and Gupta (2010) suggest that bundling may have a deterrent effect on piracy. Different from these prior works, we are more focused on restricting the patch distribution to pirates. Although we do not consider network externality effects directly, we model the benefit of releasing patches freely in that this policy diminishes hacker's incentives to exert effort.

Our work is closely related to papers by August and Tunca (2008) and Lahiri (2012). August and Tunca (2008) considers the patch restriction decision when the consumers exhibit negative network effects due to piracy. They show that a vendor benefits from restricting patch distribution to only legal users if the software is highly risky and antipiracy actions are mild, or the population's tendency to pirate is high. Lahiri (2012) considers a similar patch restriction problem but studies how the positive versus negative network externality effects impact the overall outcome. He shows that a strong positive network effect may require a less piracy-friendly and a more restrictive patching strategy (i.e., restricting patches only to legal users). Both August and Tunca (2008) and Lahiri (2012) rely on the presence of network effect to show the dominance of one policy over the other. We do not impose any network effects but, unlike them, we consider hacker's effort endogenously. We show that this endogenous variation of hacker effort can lead to one policy being dominant over the other. Our analysis on the impact of the hacker's action on the vendor's strategy is unique.

3 Model

Our setup considers a two-stage game in a software market setting. We assume that the government exerts an exogenously specified antipiracy effort of $\alpha \in [0, 1]$, where $\alpha = 0$ implies that government does not reduce pirate's welfare and $\alpha = 1$ implies that the government eliminates the pirate's welfare completely. It is reasonable to assume that government typically exerts at least some antipiracy effort. So, for the analytical portion of the paper, we set $\alpha > 0$. When numerically conducting the comparative statics, we do so over the entire range of α, allowing α to be zero.

In the first stage, a monopolistic software vendor decides the price and resources to allocate for software maintenance/patch development.[1] These decisions, however, also depend on whether or not the patch is made available to pirates. We analyze two variants of the two-stage game, each corresponding to the patch distribution policy. Eventually we compare the optimal profits obtained under the two patch distribution policies. In the second stage, a hacker and the users simultaneously make decisions. The hacker decides on the effort exerted to exploit a vulnerability in the software, and the users decide whether to buy or to pirate the software. Models such as

[1] Restrictions on patch distributions are relevant only in the context of a profit-maximizing vendor, which is what is assumed here.

ours that consider one vulnerability and one hacker have been used to derive economic insights in information security contexts before (see, for example, Kannan and Telang, 2005).

Next, we discuss the variables that model the decisions. We use a *binary* variable z to capture whether the vendor distributes patches to pirates or not. When $z = 1$, the vendor makes the patch available to all users, including the pirates; and, when $z = 0$, the patch is made available only to legal users. In our analysis, we fix z *a priori*, which requires us to analyze the two cases separately.

Corresponding to each value of z, the vendor maximizes profit by optimally setting the price and allocating resources to invest for patch development. We use p to denote the price set by the vendor for the software. We assume as is reasonable that $p \geq 0$. We capture the resources allocated for maintenance using a proxy variable that measures the probability of successfully patching the vulnerability in the software.[2] Reasonably, there is a monotonically increasing relationship between the aforementioned probability and the associated resource needs. The probability of fixing the software is denoted by x, which must lie in $[0, 1]$. We assume that when $x = 1$, the provided patch completely addresses the software vulnerability.

As mentioned above, in the second stage, both the users and the hacker make their decisions simultaneously. A key factor that affects the user's choice of whether to pirate or not is the severity of the vulnerability in the software. We model this severity as the probability that a successful exploitation of this vulnerability will render the user's system inoperable. We denote this probability by β, where $\beta \in [0, 1]$. This probability also measures the effort exerted by the hacker in attacking the user's system. Since there is a monotonic relationship between the hacker's effort and β, for simplicity of exposition, we let the hacker directly choose β. Therefore, β will be referred to as the hacker's effort. Note that we will later analyze the case when β is exogenously chosen to highlight that the economic insights change substantially in this case. An exogenous β may be a more appropriate choice for modeling performance updates.

[2]Even though the patch quality is typically realized only after the vendor allocates resources following the discovery of a vulnerability, the sequence of decisions in our model captures the key effects of reality. The decision about how much of resources to allocate for maintenance is usually made even before the software is released. Furthermore, as in our model, the consumers usually take into account a vendor's reputation for patching vulnerabilities before buying. Accordingly, we model the patch quality decision in the first stage.

3.1 Consumers and the Hacker

We model users (interchangeably, also referred to as consumers) along the lines similar to Kannan and Telang (2005). Let the consumers in the market be heterogeneous in terms of the intrinsic value they derive from the software. This consumer heterogeneity is captured by the variable $\theta \in [0, 1]$, which is assumed to be distributed uniformly. We assume that the intrinsic value of the software is θ_i^2 for a user of type θ_i. Such a characterization is consistent with the reality where the number of users having extremely large valuations is small and vice-versa. In our analysis, we deal with market-share instead of the actual number of consumers that buy the product.

The consumer utilities and the surpluses may be different depending on whether they buy or pirate the software (for example, if antipiracy efforts are strong or if patch distribution is restricted). A legal user's utility depends on the severity of the unpatched vulnerability β, and the patch quality x.[3] The term $\beta(1-x)$ models the probability with which the user's patched system is still rendered inoperable. Therefore, a consumer of type θ_i has a utility of $(1 - \beta(1 - x))\ \theta_i^2$ from buying the software. So, her expected consumer surplus from *buying* the legal software is

$$CS_b(\theta_i) = (1 - \beta(1 - x))\ \theta_i^2 - p. \tag{1}$$

We model the pirated copy of the software as an inferior but vertically differentiated substitute for the legal version. One factor that facilitates the differentiation is the government's antipiracy effort. The variable α is treated as the probability with which the pirated user may be subject to legal actions. Hence, the government's effort decreases a pirate's utility by a factor of $(1 - \alpha)$. Such a depiction of the government's antipiracy effort on a pirate's utility is similar to Baea and Choi (2006). The other factor that differentiates the legal and the pirated copies is the limit imposed by the vendor on patch distribution. When the patch is available only to legal users, a pirate is expected to suffer more from the vulnerability than when there is no such restriction. We capture this aspect by characterizing the probability that a pirate's system is rendered inoperable as $\beta(1-xz)$. Assuming a zero cost for procuring the pirated version, the consumer surplus for type

[3]We make the assumption that all users, if available, apply the patch update. This reflects the recent advancements in patch distribution and application.

θ_i from *pirating* is:

$$CS_p(\theta_i) = (1 - \beta(1 - x\ z))\ \theta_i^2\ (1 - \alpha). \tag{2}$$

Notice that α serves to vertically differentiate the legal version from the pirated version independent of the value of z, whereas β serves to vertically differentiate only when $z = 0$. Observe that consumer utility terms within the $CS_b(\theta_i)$ and $CS_p(\theta_i)$ expressions are bounded from above by one. Also, $CS_p(\theta_i)$ is non-negative. We use $\bar{\theta}$ to denote the highest consumer type that pirates the software. We make a few remarks about $\bar{\theta}$ and its properties. First, by the assumption on the consumer heterogeneity, $\bar{\theta}$ is naturally bounded between zero and one. Second, $\bar{\theta}$ exists since the consumer with $\theta_i = 0$ pirates. Third, $CS_b(\bar{\theta}) - CS_p(\bar{\theta}) \leq 0$. Fourth, a consumer of type θ_i buys the software if and only if $\theta_i > \bar{\theta}$ because $CS_b(\bar{\theta}) - CS_p(\bar{\theta})$ is a non-decreasing function of θ. In other words, we may say that there exists a $\bar{\theta}$ such that consumers of type $\theta_i \in [0, \bar{\theta}]$ pirate the product and consumers of type $\theta_i \in (\bar{\theta}, 1]$ buy the product.

Next, we focus on the decision problem faced by the hacker. Recall that β is a consequence of a hacker's strategic actions to discover and exploit the vulnerability. The notion of a strategic hacker has critical implications for the analyses, as we will discuss later. The benefit that the hacker gains from exploiting a system is characterized in a manner similar to Kannan and Telang (2005). The hacker gains are proportional to θ_i if he successfully breaks into the system of a user of type θ_i. Of course, the probabilities of breaking-in may differ for the pirated and the legal versions. This probability is simply $\beta(1 - x)$ for the legal version and $\beta(1 - xz)$ for the pirated one. Let the hacker's cost be $C(\beta)$. Since consumers of type $[0, \bar{\theta}]$ pirate and the rest buy the product, the following models the expected payoff for the hacker:

$$\pi_h(\beta) = \beta(1 - x\ z) \int_0^{\bar{\theta}} \theta\ \mathrm{d}\theta + \beta(1 - x) \int_{\bar{\theta}}^1 \theta\ \mathrm{d}\theta - C(\beta).$$

We use a logarithmic cost function for the effort: $C(\beta) = -M \log(1 - \beta)$, where $M > 0$ is an exogenous parameter. The functional form is reasonable and chosen to model that, as β increases, increasing marginal efforts are required to effect the same increase in the probability of finding a vulnerability. In particular, the limiting conditions are that the hacker incurs an infinite cost to discover the vulnerability with certainty but incurs no cost when exerting no effort. The hacker's decision problem is $\sup_{\beta\in[0,1]}\pi_h(\beta)$.

Lemma 1. *Given $\bar{\theta}$ and L, there exists an $\epsilon > 0$, such that $\sup_\beta \pi_h(\beta)$ is attained at a β^* in $[0, 1-\epsilon]$.*

Given that the hacker keeps $\beta^* < 1$, we now characterize the highest consumer type that pirates the software, denoted above as $\bar{\theta}$. Let $\kappa(\alpha,\beta,x,z) = (1-\beta)\alpha + \beta x(1-z+z\alpha)$. It is easy to check that $0 < \kappa(\alpha,\beta,x,z) \leq 1$ because $\alpha \in (0,1]$, $\beta \in [0,1)$, $x \in [0,1]$ and $z \in \{0,1\}$. Note that the difference in the utilities from buying and pirating the software for a consumer of type θ_i is $\kappa(\alpha,\beta,x,z)\theta_i^2$. Since $\kappa(\alpha,\beta,x,z) > 0$, all consumers with $\theta_i > 0$ derive a strictly higher utility from purchasing the software when compared to pirating it. In short:

$$\bar{\theta} = \min\left\{\sqrt{\frac{p}{\kappa(\alpha,\beta,x,z)}}, 1\right\}. \tag{3}$$

3.2 Vendor Profit Function

The revenue for the vendor from selling the software is $p(1-\bar{\theta})$. Let $K(x)$ denote the cost incurred for improving the patch quality. Assuming zero marginal cost for producing software, the vendor's profit is: $\pi(x,p) = (1-\bar{\theta})p - K(x)$ and his decision problem is $\sup_{(x,p)\in[0,1]\times[0,\infty)} \quad (1-\bar{\theta})p - K(x)$. We again assume a logarithmic cost function $K(x) = -L\ \log(1-x)$ for the effort, where $L > 0$ is exogenous. Observe that the cost function is reasonable. In particular, when $x = 1$, the cost is infinity, which can be interpreted to say that the vendor cannot completely secure the system. Similarly, if the vendor does not exert any effort, the cost is zero.[4]

We show that it suffices to impose $p \leq \kappa(\alpha,\beta,x,z)$ by resetting the price, whenever the above condition is violated, without altering the strategies of any of the players in any practically relevant manner. In particular, if in an equilibrium $p > \kappa(\alpha,\beta,x,z)$ then the vendor may instead set $p = \kappa(\alpha,\beta,x,z)$ and achieve the same profit because $\bar{\theta} = 1$. Further, in that case, hacker's decision does not depend on p. Therefore, the equilibrium is for all practical matters the same and although the price is reduced, no one buys the product so none of the objective values or decisions change.

[4] An earlier version of this work motivated a similar cost function in Png and Wang (2009b).

Since p may be restricted to be no more than $\kappa(\alpha, \beta, x, z)$:

$$\bar{\theta} = \sqrt{\frac{p}{\kappa(\alpha, \beta, x, z)}}. \tag{4}$$

Lemma 2. *There exists an optimal solution (x^*, p^*) to the vendor's decision problem such that $\pi(x^*, p^*) > 0$, $x^* \leq 1 - \exp\left(-\frac{1}{L}\right) < 1$, and $0 < p^* < \kappa(\alpha, \beta, x^*, z) \leq 1$. Correspondingly, $0 < \bar{\theta}^* < 1$.*

Therefore, the vendor's decision problem is $\max_{(x,p)} \left\{\pi(x, p) \;\middle|\; (x, p) \in \left[0, 1 - \exp\left(-\frac{1}{L}\right)\right] \times [0, 1]\right\}$.

4 Equilibrium Analysis

We compute the equilibria by considering the two-stage games that are obtained by setting $z = 1$ and $z = 0$ separately. For each game, we compute the Subgame Perfect Nash Equilibrium (Fudenberg and Tirole, 1991) by solving the second stage first. We solve the second stage game in a generic manner, *i.e.*, for both z values. As regards the first stage, it turns out that the equilibrium computations and analyses are straightforward when $z = 1$. However, for $z = 0$, the first stage equilibrium seems intractable. Despite this limitation, we analytically demonstrate several interesting results and supplement them with insights gained through numerical analyses.

We first focus on the second stage which constitutes a simultaneous game between a hacker and consumers where the players decide respectively on the level of hacking effort and whether or not to buy the software. In the proof, we use a technical property of certain types of functions, that we refer to as *single-crossing functions.*[5]

Definition 3. *A function $h(u) : \mathbb{R} \mapsto \mathbb{R}$ single crosses another function $g(u) : \mathbb{R} \mapsto \mathbb{R}$ from above (respectively, below) if, for all $u'' > u'$, $h(u') - g(u') \leq 0$ implies that $h(u'') - g(u'') < 0$ (resp., $h(u') - g(u') \geq 0$ implies that $h(u'') - g(u'') > 0$).*

Intuitively, the above definition captures the property that the graphs of two functions intersect at no more than one point. First observe that if, as per Definition 3, $h(u)$ single crosses $g(u)$ from above then $h(u)$ and $g(u)$ can intersect at at most one point. To see this, assume that $h(u') = g(u')$

[5] Note our definition is different from the manner in which the single crossing property is associated with Spence-Mirlees condition in economics.

and $h(u'') = g(u'')$. Without loss of generality let $u' < u''$. Then, the definition above requires that $h(u'') < g(u'')$ which is in contradiction of our hypothesis. Next, note that if $h(u) - g(u)$ is a continuous function, $h(u) - g(u)$ is positive and negative for some values of u, and $h(u)$ does not equal $g(u)$ at more than one point then $h(u)$ single crosses $g(u)$ from above or from below. Assume that $h(u)$ equals $g(u)$ only if $u = \bar{u}$. Then, because of continuity, and by exchanging g and h if necessary, we may assume without loss of generality that $h(u) - g(u) > 0$ for $u < \bar{u}$ and $h(u) - g(u) < 0$ for $u > \bar{u}$. Then, we show that $h(u)$ single-crosses $g(u)$ from above. Consider any u' such that $h(u') - g(u') \leq 0$. Then, $u' \geq \bar{u}$ and therefore, for all $u'' > u'$, $h(u'') - g(u'') < 0$. The proof for the second stage solution exploits the following technical lemma which captures a simple property of the single crossing functions.

Lemma 4. *Assume for every $i \in I$, $h(u)$ single crosses $g_i(u)$ from above (resp. below), then $h(u)$ single crosses $\max_{i \in I} g_i(u)$ from above (resp. below).*

Now, we are ready to prove the uniqueness of the solution to the second-stage game.

Lemma 5. *The equilibrium second stage variables are obtained by solving the simultaneous equations involving (3) and*

$$\beta \quad = \quad \max\left\{1 - \frac{2M}{1 - x\left(1 - (1-z)\bar{\theta}^2\right)}, 0\right\} \tag{5}$$

The resulting solution $\{\bar{\theta}^*, \beta^*\}$ is unique, $\beta^* < 1$, and also, $\forall\ M \in [\frac{1}{2}, \infty]$, $\beta^* = 0$.

We make significant use of the uniqueness of $\{\bar{\theta}^*, \beta^*\}$ in deriving our results. We remark that although the techniques used in the proof of Lemma 4 depend on the functional forms, they do so primarily by exploiting the strict monotonicity of $\sqrt{\frac{p}{\kappa(\alpha,\beta,x,z)}}$ and $1 - \frac{2M}{1-x(1-(1-z)\bar{\theta}^2}$ and the fact that they satisfy the single-crossing property defined above. We will build on these aspects later.

The lemma shows that when $M \geq \frac{1}{2}$, there is no hacker effort (i.e., $\beta^* = 0$) for either z. Consequently, (3) is independent of z and as a result the patch quality x or price p does not depend on z either. So, the prices, profit, welfare outcomes are the same and the policy comparison becomes uninteresting. To focus on the interesting cases, we restrict M to be less than 1/2 hereafter. The following subsections deal with the first stage decisions for each patch distribution policy, *i.e.*, value of z, separately.

4.1 Patch available to all users ($z = 1$)

In this section, the vendor makes the patch available to everybody. Setting $z = 1$, we obtain from (4) and (5) that:

$$\{\beta^*, \quad \bar{\theta}^*\} = \left\{\max\left\{1 - \frac{2M}{1-x}, 0\right\}, \quad \min\left\{\sqrt{\frac{p}{\alpha(1-\beta^*(1-x))}}, 1\right\}\right\}. \tag{6}$$

Observe that since patches are made available to everybody, the hacker's payoff is independent of the share of the market that buys the product. Consequently, β^* expression is independent of $\bar{\theta}^*$. To simplify the notation we denote $(\beta^*, \bar{\theta}^*)$ as (B, Θ) in the remainder of this section. Then, Θ^* and B^* respectively denote the untapped market and the hacker effort at the price p^* and patch quality x^* that yields maximum profit for the vendor. In the following π_v^* denotes the maximum profit of the vendor. Using the above expressions in the vendor's decision problem, we compute the equilibrium:

Proposition 6. *The equilibrium market share for the vendor is $\frac{1}{3}$; the corresponding decisions for the vendor and the hacker are:*

$$\{p^*, x^*, B^*\} = \begin{cases} \left\{\frac{1}{9}(8M\alpha), 0, 1-2M\right\} & L \geq \frac{4\alpha}{27} \\ \left\{\frac{1}{9}(8M\alpha + 4\alpha - 27L), \frac{4\alpha - 27L}{4\alpha}, 1 - \frac{8\alpha M}{27L}\right\} & \frac{4\alpha}{27} \geq L > \frac{8\alpha M}{27} \\ \left\{\frac{4\alpha}{9}, 1-2M, 0\right\} & \frac{8\alpha M}{27} \geq L > 0; \end{cases}$$

and the vendor profits are

$$\pi_v^* = \begin{cases} \frac{8M\alpha}{27} & L \geq \frac{4\alpha}{27} \\ \frac{8M\alpha}{27} + \frac{4\alpha}{27} - L + L\log\left(\frac{27L}{4\alpha}\right) & \frac{4\alpha}{27} \geq L > \frac{8\alpha M}{27} \\ \frac{4\alpha}{27} + L\log(2M) & \frac{8\alpha M}{27} \geq L > 0. \end{cases}$$

Notice from Proposition 6 that there are three regions where an equilibrium may occur. When the cost for the quality of patch is sufficiently high, i.e., $L \geq \frac{4\alpha}{27}$, the vendor does not have an incentive to develop a patch, *i.e.*, $x^* = 0$, and the equilibrium is straightforward. When the cost of the hacker's effort is large relative to vendor's effort $M > \frac{27L}{8\alpha}$ (which is same as, $\frac{8\alpha M}{27} > L$), the hacker is completely driven out *i.e.*, $B^* = 0$. For an intermediate L, both x^* and B^* are non-zero. The next proposition gives insights on how the equilibrium varies with problem parameters.

Proposition 7. *We observe the following properties with respect to the equilibrium outcomes:*

1. $\frac{\partial p^*}{\partial \alpha} > 0$ *and* $\frac{\partial x^*}{\partial \alpha} \geq 0$.

2. $\frac{\partial p^*}{\partial L} \leq 0$ *and* $\frac{\partial x^*}{\partial L} \leq 0$.

3. $\frac{\partial p^*}{\partial M} \geq 0$ *and* $\frac{\partial \pi_v^*}{\partial M} \geq 0$

4. *If* $B^* > 0$, $\frac{\partial x^*}{\partial M} = 0$; *otherwise,* $\frac{\partial x^*}{\partial M} < 0$.

5. $\frac{\partial B^*}{\partial \alpha} \leq 0$, *and* $\frac{\partial B^*}{\partial L} \geq 0$.

The first three results in the above proposition are fairly intuitive. When patches are distributed to pirates and non-pirates, the only source of differentiation is the governmental effort α. Therefore, an increase in α increases the vertical differentiation. It enables p to increase. In that case, the marginal benefit from the patch quality also increases and, therefore, x increases. Next, consider the second result. When L increases, the marginal cost of x increases and so, x^* decreases. Consequently, since the consumer utility decreases, p^* decreases. The third result shows that when the hacker decreases the effort because of a higher M, the vendor is able to extract more surplus from the legal users and improve his profits.

The conclusion of the fourth result varies depending on whether the hacker effort B^* is zero or not. When $B^* = 0$, the vendor only needs to maintain a patch quality that is sufficient to keep the hacker out. As M increases, the hacker is dis-incentivized to exert effort anyway and, hence, the vendor's effort to keep the hacker out decreases. Interestingly, when the hacker's effort is non-zero, the patch quality x^* chosen by the vendor is independent of the cost incurred by the hacker. Observe that, when hacker exerts a non-zero effort, the software price varies linearly in M and x, making the patch quality decision independent of hacker's effort. In other words, the vendor chooses a price that absorbs the damage caused by the hacker in such a way that the patch quality depends only on L and vertical differentiation due to anti-piracy effort, α.

The fifth result in the proposition can be explained as a consequence of the first one. As α increases (i.e., the antipiracy effort intensifies), the vendor improves the patches that are delivered.

This increase in patch quality decreases the marginal payoff that hacker obtains. Hence, the decrease with respect to α. A similar reason explains why B^* increases as L increases.

4.2 Patch available to legal users only ($z = 0$)

In this section, we consider the case where the vendor makes the patch available only to the *legal users*. Set $z = 0$ in (3) and (5) to see that the second stage equilibrium is obtained as a solution of:

$$\{\beta^*, \bar{\theta}^*\} = \left\{ \max\left\{1 - \frac{2M}{1 - x(1 - \bar{\theta}^{*2})}, 0\right\}, \quad \min\left\{\sqrt{\frac{p}{\alpha - \beta^*(\alpha - x)}}, 1\right\}\right\}. \tag{7}$$

Earlier when $z = 1$, β^* was independent of $\bar{\theta}^*$ and as a result, the vendor's decision problem was relatively straightforward to solve in closed-form. In the above equation, however, because the expression for β^* involves $\bar{\theta}^*$ and vice-versa, the vendor's decision problem does not seem to have a simple closed-form solution. This makes the ensuing analysis somewhat more complicated.

Given α, x, and M, we show that we may construct an invertible relation between p and $\bar{\theta}^*$. First observe that Lemma 4 shows that, given a p, there is a unique $\bar{\theta}^*$ that solves (7). Now, we consider the reverse direction and obtain p as a function of $\bar{\theta}^*$. Given a $\bar{\theta}^* < 1$, p can be easily computed using (7) since β^* is a function of $\bar{\theta}^*$, M, and x which are all available. Now, consider the case when $\bar{\theta}^* = 1$. Clearly, the price is not uniquely determined in this case. However, we recall that when $\bar{\theta}^* = 1$, setting $p = \kappa(\alpha, \beta, x, 1)$ does not change the equilibrium in any practically relevant way (see discussion prior to Lemma 2). Furthermore, the vendor sets the price in a way such that $\bar{\theta}^*$ is strictly less than 1. Therefore, we set $p = \kappa(\alpha, \beta^*, x, 1)$ when $\bar{\theta}^* = 1$, realizing that there is no equilibrium at which $p \geq \kappa(\alpha, \beta^*, x, 1)$. It follows that the mapping between p and $\bar{\theta}^*$ is invertible.

The primary advantage of the invertible mapping between p and $\bar{\theta}^*$ is that it allows us to interpret the vendor's action as choosing $(x, \bar{\theta}^*)$ instead of (x, p). This is particularly important since expressing $\bar{\theta}^*$ as a function of p is unwieldy, whereas the inverse mapping is easy to express. As before, we denote $(\beta^*, \bar{\theta}^*)$ as (B, Θ) in the remainder of this section. We have also restricted the domain of x based on Lemma 2. With these transformations, the vendor's problem reduces to:

$$\max_{(x,\Theta)\in[0,1-\exp(-\frac{1}{L})]\times[0,1]} \pi_v'(x,\Theta) = \begin{cases} (1-\Theta)\Theta^2\alpha + L\log(1-x) & \text{if } x > \frac{1-2M}{(1-\Theta^2)} \\ (1-\Theta)\Theta^2\left((\alpha-x)\dfrac{2M}{(1-(1-\Theta^2)x)}+x\right) & \text{otherwise} \\ \quad +L\log(1-x) & \end{cases} \tag{8}$$

Observe that the condition $x > \frac{1-2M}{(1-\Theta^2)}$ corresponds to the case when $B = 0$. Further, when $x = \frac{1-2M}{(1-\Theta^2)}$, evaluating any of the expressions on the right-hand-side yields the same value. We denote the optimal solution to (8) as (x^*, Θ^*), the corresponding price as p^* and the equilibrium effort of the hacker as B^*.

Lemma 8. $x^* \not> \frac{1-2M}{1-\Theta^{*2}}$. *If* $B^* = 0$, *then* $x^* = \frac{1-2M}{1-\Theta^2}$ *and* $\Theta^* \leq \frac{2}{3}$.

Lemma 8 shows that, when the hacker does not exert effort, *i.e.* $B^* = 0$, the vendor will choose the minimum patch quality that is needed to keep the hacker from exerting any effort. The lemma also allows us to further restrict the domain of x to $\left[0, \frac{1-2M}{1-\Theta^2}\right]$. This reformulates (8) into the following optimization problem:

$$\begin{aligned} \max_{(x,\Theta)} \quad & L\log(1-x) + (1-\Theta)\Theta^2\left(\frac{2M(\alpha-x)}{(1-(1-\Theta^2)x)}+x\right) \\ & 0 \leq x \leq \min\left\{1-\exp\left(-\frac{1}{L}\right), \frac{1-2M}{1-\Theta^2}\right\} \\ & 0 \leq \Theta \leq 1 \end{aligned} \tag{9}$$

Since $x \leq 1-\exp\left(-\frac{1}{L}\right) < 1$, it follows that the objective function is continuous over the feasible region, which in turn is compact. Therefore, it follows from Weierstrass Theorem that there exists an optimal solution to (9), a consequence which also follows from Lemma 2.

This specification also makes it clearer why it is hard to solve the vendor's problem in closed-form. Although it might appear at first glance that the intractability is due to the logarithmic cost function of x, it is in fact still difficult to optimize the objective of (9) with respect to Θ when x is fixed. To see this, fix x in (9) and maximize vendor profit with respect to Θ. Then, the first order optimality condition of the objective function with respect to Θ yields a fifth order polynomial equation. We show in the next example that this observation is at the heart of why vendor's problem is difficult to solve in closed-form.

Example 9. *According to Abel-Rufini's theorem, there does not exist a generic algebraic solution to polynomial equations of degree five or higher. In fact, Galois' Theory sharpens Abel-Rufini's theorem by showing that a polynomial is solvable in radicals if and only if the associated Galois group is solvable. When one chooses* $M = \frac{1}{11}$, $\alpha = \frac{1}{2}$, *and* $x = \frac{1}{4}$, *the first order optimality condition with respect to* Θ *reduces to the solution of*

$$33\Theta^5 - 22\Theta^4 + 206\Theta^3 - 132\Theta^2 + 369\Theta - 246 = 0. \tag{10}$$

Since the associated Galois group is not solvable, the roots of this polynomial equation cannot be expressed exactly with radicals. We are particularly interested in the root $\hat{\Theta}$ *that lies in* $[0.66, 0.67]$. *It can be verified using Sturm's theorem that there is exactly one such root. Observe that* $\frac{\partial \pi_v(x,\Theta)}{\partial \Theta}$ *does not depend on* L. *On the other hand, a change in* L *impacts the optimal patch quality* x^*. *We now show that there exists an* $L \in \left[\frac{1}{10}, \frac{1}{11}\right]$ *such that* $\left(x = \frac{1}{4}, \Theta = \hat{\Theta}\right)$ *satisfies the first order optimality conditions for (9). First, let* $L = \frac{1}{10}$. *It is easy to verify that* $\left.\frac{\mathrm{d}\pi_v(x,\Theta)}{\mathrm{d}x}\right|_{x=\frac{1}{4}}$ *has the same sign as* $-165\Theta^7 + 165\Theta^6 - 750\Theta^5 + 728\Theta^4 - 1245\Theta^3 + 1113\Theta^2 - 198$ *which, by Sturm's theorem, is negative for all* $\Theta \in [0.66, 0.67]$. *Therefore,* $\left.\frac{d\pi_v(x,\Theta)}{\mathrm{d}x}\right|_{x=\frac{1}{4},\Theta=\hat{\Theta}} < 0$ *when* $L = \frac{1}{10}$. *Similarly, when* $L = \frac{1}{11}$, *it can be shown that* $\left.\frac{d\pi_v(x,\Theta)}{\mathrm{d}x}\right|_{x=\frac{1}{4},\Theta=\hat{\Theta}} > 0$. *Since* $\left.\frac{d\pi_v(x,\Theta)}{\mathrm{d}x}\right|_{x=\frac{1}{4}}$ *is a continuous function of* L, *there exists an* L *such that* $\left(\frac{1}{4}, \hat{\Theta}\right)$ *satisfies first order optimality conditions for (9). In fact, it can be verified numerically (within tolerances) that this solution is the optimal decision for the vendor. Numerical verification here is reasonable since it can be shown that there are a finite number of points that satisfy the local optimality conditions. This was done by verifying, using Sturm's theorem, that the leading terms of a Groebner basis of the polynomials obtained from the first order optimality conditions do not vanish in the specified range of* L. *Since we have argued that* $\hat{\Theta}$ *cannot be expressed exactly using radicals, this example shows that there is little hope for closed-form solutions to the vendor's problem when patches are restricted to legal users.*

Although we shall not be able to express the optimal solution in closed-form, we expose various interesting properties of the solution. Recall that, by Lemma 2, we may restrict $\Theta \in (0, 1)$. For a given $\Theta \in (0, 1)$, we denote the optimal patch quality as $x^*(\Theta)$. Similarly, given x, we denote the optimal untapped market as $\Theta^*(x)$. By Weierstrass Theorem, it can be easily verified that these optimal solutions exist.

Theorem 10. *1. Given $\Theta \in (0,1)$, $x^*(\Theta)$ is unique and decreasing in M. For $L \geq \frac{4}{27}$, $x^* = 0$.*

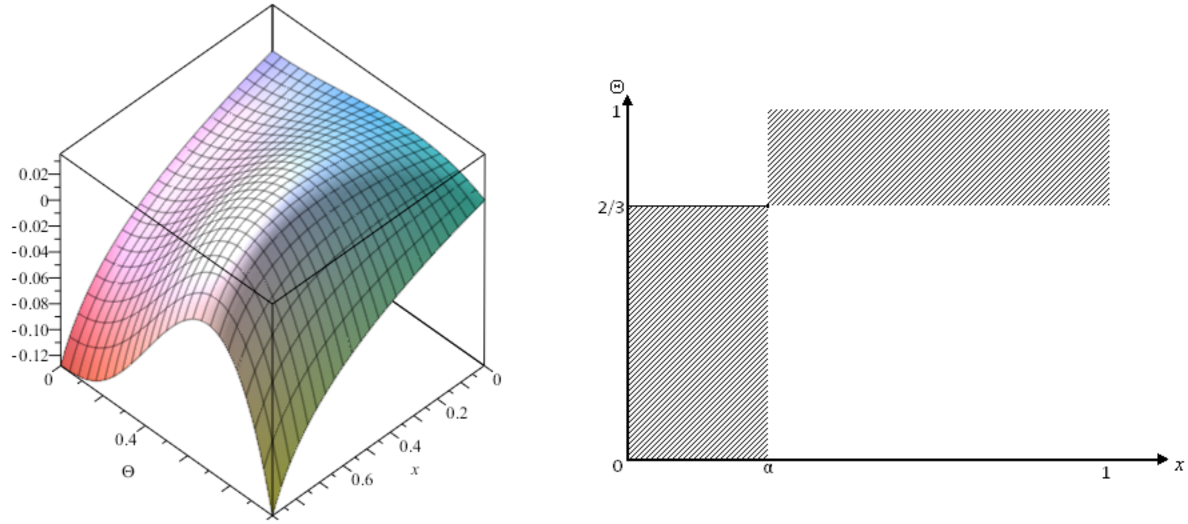
2. Let $\Theta' \in \Theta^*(x)$. If $x = \alpha$ then $\Theta' = \frac{2}{3}$. If $x > \alpha$, $\Theta' > \frac{2}{3}$. Finally, consider $x < \alpha$. If $x \leq \frac{9}{5}(1-2M)$ then $\Theta' = \frac{2}{3}$. Otherwise, $\Theta' < \frac{2}{3}$.

3. $\exists\{\alpha, L\}$ such that for $M = \hat{M}$, $x^* = \alpha$ and $B^* \neq 0$. Further, there exists an ϵ such that if $M = \hat{M} + \epsilon$ (resp. $M = \hat{M} - \epsilon$) then $x^* < \alpha$ (resp. $x^* > \alpha$) and $B^* \neq 0$.

Arguably, Theorem 10 is the most important result that enables us to derive analytical insights from the model. The reader may find the details of the proof interesting. In particular, consider Point #3 of Theorem 10. Although it is reasonably straightforward to show that $\left(\alpha, \frac{2}{3}\right)$ satisfies local optimality conditions for the vendor's optimization problem for some settings of the parameters, this is not sufficient to prove Point #3 because the vendor's objective is not concave (as illustrated in Figure 1(a)). Therefore, it is necessary to check global optimality of this decision for the vendor. The main trick in the proof is the construction of a nonconvex overestimator for the vendor's objective over the relevant region for which we establish using semialgebraic geometry tools that $\left(\alpha, \frac{2}{3}\right)$ is the global optimum. Then, because the overestimator exactly estimates the vendor's objective at $\left(\alpha, \frac{2}{3}\right)$, the result follows.

Using Point #1, L can be bounded for numerical computations. This result in conjunction with Proposition 6 shows that if $L \geq \frac{4}{27}$ then $x^* = 0$ regardless of the patch distribution policy. Then, Θ, B, p are also independent of z because the consumer surplus does not depend on z. The comparisons are therefore uninteresting in this case and we restrict $L \leq \frac{4}{27}$ hereafter.

The insights from the above theorem are interesting in several ways. First, observe that Point #2 shows that the optimal solutions have an interesting and somewhat counterintuitive, structure. In particular, when x^* is large, the market tapped is small and vice-versa. This is because when the vendor exerts significant effort towards improving the patch quality, it also increases the price significantly so that the market share is reduced. This structure can also be seen from Figure 1(b). Second, as mentioned earlier, the vendors' argument for limiting the availability of patches is that the patch restrictions encourage more users to buy legal versions. In contrast, Points #2 and #3



(a) Concavity of the vendor profit

(b) Solution space $\Theta \times x$: The marked area is where the solution can fall in

Figure 1: Figures to explain Theorem 10

in Theorem 10 show that sometimes (specifically $M < \hat{M}$) fewer consumers buy the software when patch is restricted in distribution. This result shows that the vendor does not limit patches simply to target the pirates but may strategically restrict patches to improve profits by extracting more surplus from the legal users.

Point #3 leads to the following theorem:

Theorem 11. *For certain values of* (α, M, L)*, the vendor profit increases with* M *and, for certain others, it decreases. In other words,* $\frac{\partial \pi_v^*}{\partial M}$ *is sometimes positive and other times negative.*

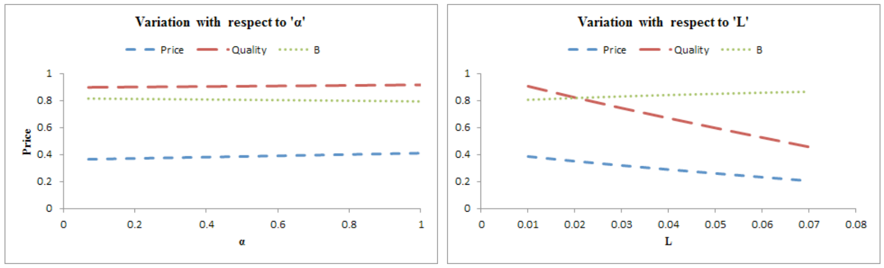
At the outset, one might expect that the hacker effort decreases vendor profit because an increased hacker effort decreases consumer surplus which the vendor uses to extract a profit. This obvious effect will henceforth be referred to as the *adverse effect* of hacker's effort on vendor's profit. However, Theorem 11 shows that that an increase in hacker's effort may allow the vendor to extract more profit. This happens because an increase in B can help vertically differentiate the legal version from the pirated version of the software. When the patch is only distributed to legal users, the hacker is incentivized to exert more effort, which diminishes a pirate's utility more than it reduces a legal user's utility. In effect, the hacker makes the legal version more valuable for the consumers, which allows the vendor to charge a higher price to generate more profit. We refer to this effect as the *countervailing effect* of hacker's effort.

We now explain, in an intuitive manner, the primary effects that drive some of the results derived in Theorems 10 and 11. First observe that the vendor's profit increases when more users purchase a legal copy of the software, *i.e.*, when Θ reduces. Since $\Theta = \sqrt{\frac{p}{\alpha - B(\alpha - x)}}$, the effect of hacker's effort, B, on Θ and hence on vendor's profit depends on the sign of $\alpha - x^*$. More specifically, if $\alpha > x^*$, *i.e.*, when government exerts significant antipiracy effort, Theorem 10 shows that $\Theta^* < \frac{2}{3}$, *i.e.*, vendor's share is more than $\frac{1}{3}$. Observe that hacker's effort decreases monotonically with increase in his cost parameter M. Then, a decrease in hacker's effort or, equivalently, an increase in M, results in a decrease in Θ and thus an increase vendor's profit. However, when $\alpha < x^*$, *i.e.*, government does not exert significant antipiracy effort, an increase in hacker's effort or equivalently a reduction in M, leads to a decrease in Θ when p is fixed and therefore increases vendor's profit. Interestingly, despite this tendency of Θ to increase for a given price, we show in Theorem 10 that the vendor sets a price such that $\Theta^* > \frac{2}{3}$ and the resulting market share is strictly less than $\frac{1}{3}$. In other words, the vendor is able to exploit the countervailing effect of hacker and takes advantage of the vertical differentiation between legal and pirated software arising from hacker's efforts.

4.2.1 Numerical Analysis

Note that all the relevant parameters are bounded: $\alpha \in [0, 1]$ by model specification; Lemma 5 and Theorem 10 provide bounds on M and L, respectively; the endogenous variables Θ, B, and x are each in the range $[0, 1]$ by model construction; Lemma 2 further restricts x and p; and the same lemma establishes existence of an optimal solution to the vendor's problem. Since vendor's profit is continuous (see Equation 9), we explore the solution space by constructing a grid of points within the above bounds.

As before, we denote the optimal price by p^*, patch quality by x^*, vendor profit by π^*, untapped market by Θ^*, and hacker effort by B^*. Unless specified otherwise, it is assumed that these solutions correspond to the case when the patches are restricted in distribution to the legal users. Figure 2 shows how p^* and x^* vary with respect to L and α. The variations are similar to those when $z = 1$ (see Proposition 7) and so are the explanations. Next, consider the variation of x^* and π^* with respect to M. When $x^* < \alpha$ (see Figures 3(d)-3(f)), the corresponding profits in Figures 3(a)-3(c) increase with M, which one may recall as the adverse effect described after Theorem 11. However,



(a) Variation with respect to α ($M = 0.05$, $L = 0.01$)

(b) Variation with respect to L ($\alpha = 0.5$, $M = 0.05$)

Figure 2: Effect of Antipiracy Effort and Cost of Patch Quality.

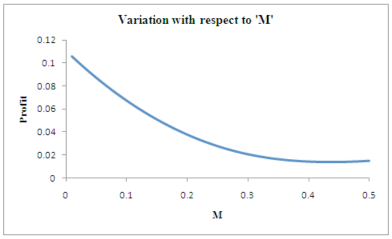
when $\alpha < x^*$, it can be observed in the same figures that profits decrease with an increase in M because of the countervailing effect.

Our numerical analysis also shows that patch quality does not increase as M increases (see Figures 3(d), 3(e), and 3(f)). In some cases, the vendor does not exert any effort on patch quality, *i.e.*, x^* equals zero (see Figure 3(f)). In contrast, in other settings, the vendor provides a sufficient patch quality to drive the hacker out *i.e.*, $B^* = 0$). In this case, x^* is maintained at the level necessary to keep the hacker marginalized. In Figure 3(e), we observe a kink at the value of M beyond which x^* is maintained at the minimum level necessary to keep the hacker from exerting effort $B^* = 0$.

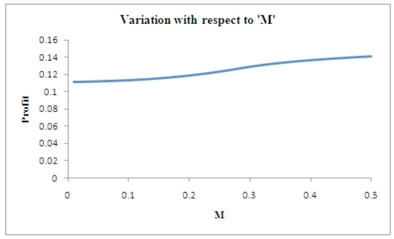
Next, consider the variation of price and market-share with M. When the countervailing effect is dominant, the vendor appears to always decrease p^* as M increases. Since both the price as well as the patch quality can influence the type of consumer that is indifferent, the market share variations are not monotonic. These can be seen in Figures 3(k), 3(h), 3(l), and 3(i). In contrast, when the adverse effect is dominant, both the price and the market-share vary non-monotonically with M. When the adverse effect is dominant, for the lower range of M, the price cut is insufficient to make up for the decreased differentiation and, the market share continues to fall. However, for slightly larger M, the price cut makes up for the decreased differentiation and the market-share improves. These can be seen in Figures 3(j), 3(g), 3(l), and 3(i).

We make the following observations based on the numerical analysis:

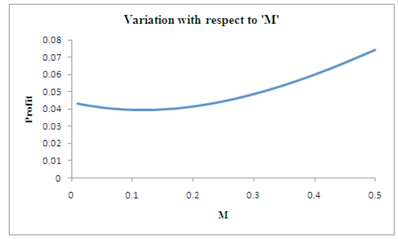
Numerical Observation 12. • *The optimal patch quality decreases as M increases.*



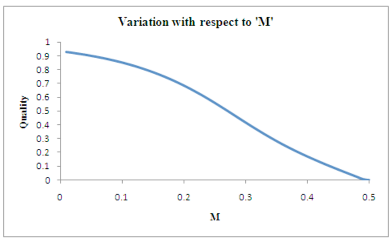
(a) Optimal Profit: $\alpha = 0.10$, $L = 0.01$



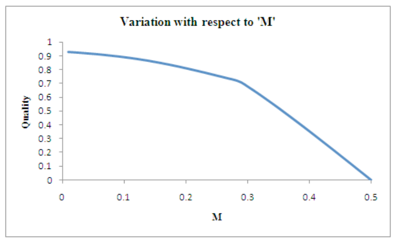
(b) Optimal Profit: $\alpha = 0.95$, $L = 0.01$



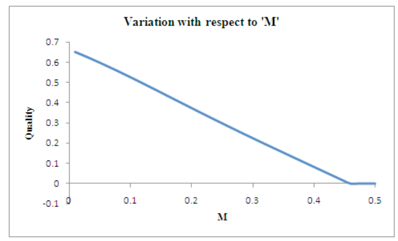
(c) Optimal Profit: $\alpha = 0.50$, $L = 0.05$



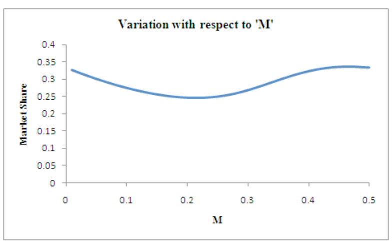
(d) Patch Quality: $\alpha = 0.10$, $L = 0.01$



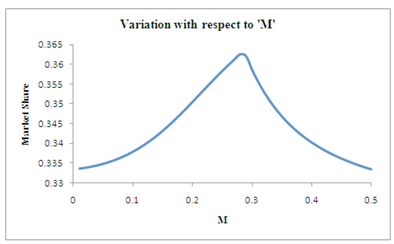
(e) Patch Quality: $\alpha = 0.95$, $L = 0.01$



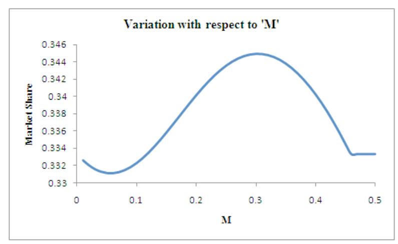
(f) Patch Quality: $\alpha = 0.50$, $L = 0.05$



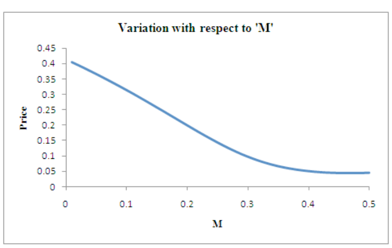
(g) Optimal Market Share: $\alpha = 0.10$, $L = 0.01$



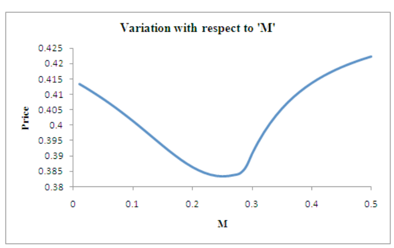
(h) Optimal Market Share: $\alpha = 0.95$, $L = 0.01$



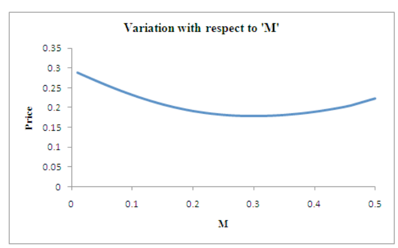
(i) Optimal Market Share: $\alpha = 0.50$, $L = 0.05$



(j) Optimal Price: $\alpha = 0.10$, $L = 0.01$



(k) Optimal Price: $\alpha = 0.95$, $L = 0.01$



(l) Optimal Price: $\alpha = 0.50$, $L = 0.05$

Figure 3: Effect of M on Optimal Price, Profit, Market Share, and Patch Quality

- *Within a certain range of L, whenever $M = \hat{M}_{x=\alpha} = \frac{(9-5\alpha)(4-4\alpha-27L)}{72(1-\alpha)}$ (resp. $M = \hat{M}_{x=0} = \frac{9(4-27L)}{8(9-5\alpha)}$), the equilibrium is attained when $x^* = \alpha$ (resp. $x^* = 0$) and $\Theta^* = \frac{2}{3}$.*

Observe that a partial proof of the second numerical observation can be obtained by adapting the proof of Theorem 10. To see this, observe that the settings of the parameters in the proof: $\alpha = \frac{1}{2}$, $L = \frac{1}{18}$, and $M = \frac{13}{144}$ satisfy the relation $M = \frac{(9-5\alpha)(4-4\alpha-27L)}{(72(1-\alpha))}$. Whenever M satisfies the above relation, $\left(\alpha, \frac{2}{3}\right)$, it is easy to check that $\left(\alpha, \frac{2}{3}\right)$ satisfies the local optimality conditions because, at this point, the partial derivative of $\pi_v(x, \Theta)$ is zero with respect to both variables. Moreover, it was shown that the derivative of $\pi_v(x^*(\Theta), \Theta)$ is never zero elsewhere. Since the coefficient of the associated polynomials are continuous functions of α, L, and M, it follows that small perturbations of these parameters do not alter the difference of the number of sign changes of the Sturm Sequence over the intervals used in the proof since the roots do not occur at the end-points. Therefore, even when small perturbations are made to $\{\alpha, L, M\}$ in such a way that $M = \frac{(9-5\alpha)(4-4\alpha-27L)}{(72(1-\alpha))}$, the optimal solution remains $\left(\alpha, \frac{2}{3}\right)$.

5 Comparative Statics

This section focuses on comparing the equilibrium outcomes and welfare implications under the two patch distribution policies, *i.e.*, with $z = 1$ versus $z = 0$. The first subsection compares the equilibrium values of p^*, x^*, B^*, and Θ^*, whereas the second one analyzes the corresponding social welfare generated. Like in the previous section, we establish some of the results analytically while others are found via numerical computations.

As before, we deal with $\alpha \neq 0$ in the analytical portion of the subsections below. When dealing with $\alpha = 0$ in the numerical portion, the discussion that follows will be relevant. The consumer utilities from buying and pirating can be the same when $\alpha = 0$. If the consumer utilities from buying and pirating are the same then either $p = 0$ or $p > 0$ and no one buys the software. In either case, the vendor does not have any revenue or profit. So, the market may not exist in such cases. If $z = 1$ then $\alpha = 0$ is necessary and sufficient for the utilities to be the same (and the market to not exist). If $z = 0$ then $\alpha = 0$ is necessary but not sufficient. Here, in addition to $\alpha = 0$, the versions cannot be differentiated only if either $B^* = 0$ or $x^* = 0$.

Example 13. *Let $z = 0$, $\alpha = 0$, $M = 0.3$, and $L = 0.01$. Then, we numerically find that $p^* \approx 0.076$, $\Theta^* \approx 0.79$, $x^* \approx 0.44$, and $B^* \approx 0.28$. In contrast when $M = 0.47$, the market does not exist.*

5.1 Comparison of p^*, x^*, π^*, B^*, and Θ^*

We first compare vendor's profit under the different patch distribution policies. To fix notation, we denote the value of z as a subscript. For example, p_0^* (resp. p_1^*) is the optimal price when patches are restricted in distribution (resp. freely available).

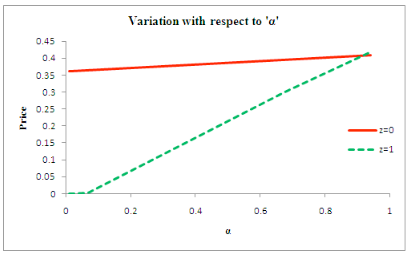
Proposition 14. *There exists an $\tilde{\alpha}$ such that, for all $\alpha \in [\tilde{\alpha}, 1]$, the equilibrium vendor profit is strictly higher when patches are released to everyone compared to when patches are restricted in distribution.*

The above result shows that the vendor may have a larger profit by releasing the patch to everyone in comparison to adopting the restricted patch distribution policy. This seemingly counter-intuitive result can be explained as follows. When government puts a large antipiracy effort, there is not much value for pirates and, as a result, the product is already vertically differentiated. Unpatched pirated copies increase the incentive of the hacker to exert effort and this action takes away consumer surplus from legal users as well which the vendor could have extracted as profit. Instead, the vendor offers the patches to everyone leaving anti-piracy efforts to the government, thereby discouraging hackers and extracting the higher surplus from its consumers.

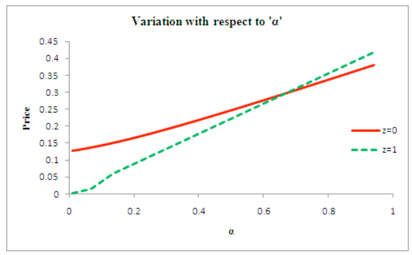
We next show that the vendor will choose to distribute the patches to all the users only if the hacker's action is endogenous. For this purpose, we diverge from the model specification proposed in this paper and regard β as exogenous.

Proposition 15. *If β is exogenously specified (not decided by the hacker as a strategic variable) the equilibrium is such that $\bar{\theta} = \frac{2}{3}$ regardless of the patch distribution policy. Further, it is never strictly optimal for the vendor to release the patch to everyone.*

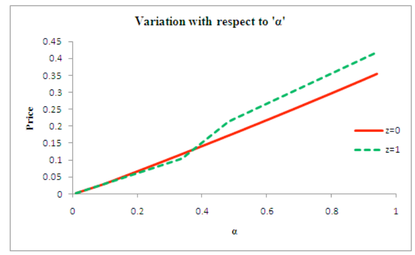
As discussed before, we can consider exogenous β as modeling the case of performance updates because, in this setting, hacker's action does not depend on the quality of the update. For such updates, if the vendor restricts the distribution of the patch, while maintaining the same price and patch quality, then more consumers have an incentive to purchase the legal copy of the software since performance updates increase the software's utility. To take advantage of this fact, the vendor increases the price, still serves the same fraction of the market, and earns a higher profit. It follows that it is never strictly better for the vendor to release performance updates freely. We remark that the analytical difficulty of deriving the equilibrium solution is significantly reduced when β is exogenous. In this case, closed-form solutions are amenable because the second-stage game is simple. Moreover, the structure of the solution is also less rich when β is exogenous. In particular, with exogenous β, the vendor targets $\frac{1}{3}$rd of the market regardless of the patch distribution strategy. Notwithstanding, when β is endogenous and patches are restricted in distribution, the vendor may target a smaller or larger market share in the equilibrium strategy.



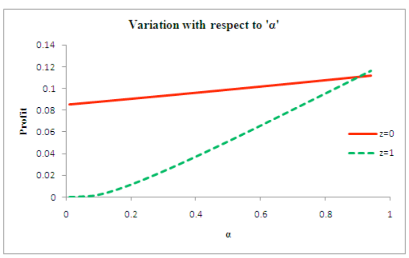
(a) Price ($M = 0.05$, $L = 0.01$)



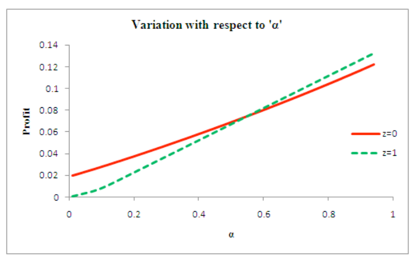
(b) Price ($M = 0.25$, $L = 0.01$)



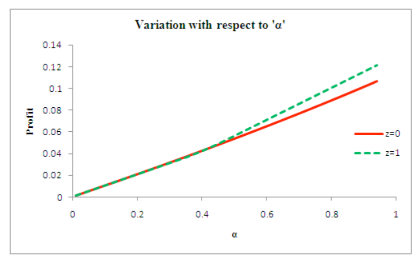
(c) Price ($M = 0.35$, $L = 0.05$)



(d) Profit ($M = 0.05$, $L = 0.01$)



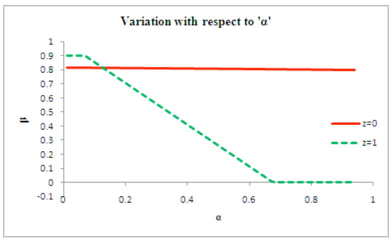
(e) Profit ($M = 0.25$, $L = 0.01$)



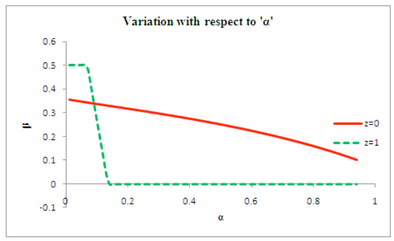
(f) Profit ($M = 0.35$, $L = 0.05$)

Figure 4: Comparative Statics - Effect of α.

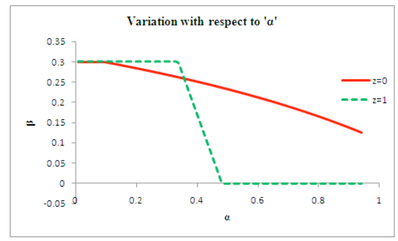
We now revert back to the case of endogenous hacker action, which is more appropriate for security updates, and analyze the model using numerical computations. Figure 4 shows the variation of profits and prices with α. For low values of α, the vendor prefers to restrict patches and vice-versa. When $\alpha = 0$, the market cannot exist with unrestricted patches ($z = 1$) but the vendor may be profitable for $z = 0$ by exploiting the countervailing effect (see Example 13 for a specific scenario). This tendency to exploit the hacker's effort to differentiate the two versions and obtain higher profits continues even for slightly larger α values. With an increase in α, the incentive of the vendor to employ the hacker's effort to differentiate the versions decreases. Consequently, vendor continues to restrict the patch until a threshold α and thereafter releases the patch freely. The threshold α decreases as M increases. This can be explained because, as M increases, the hacker's effort reduces and the vendor's reliance on the countervailing effect decreases. The price variations in the figures can also be similarly explained.



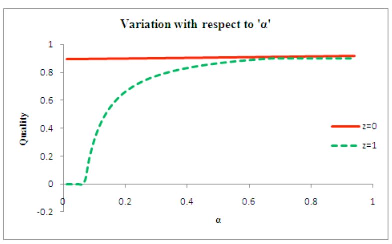
(a) Hacker's Effort ($M = 0.05$, $L = 0.01$)



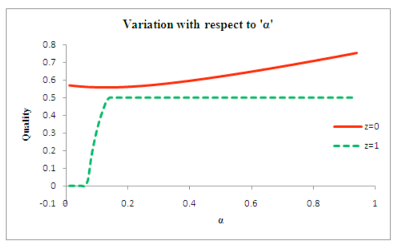
(b) Hacker's Effort ($M = 0.25$, $L = 0.01$)



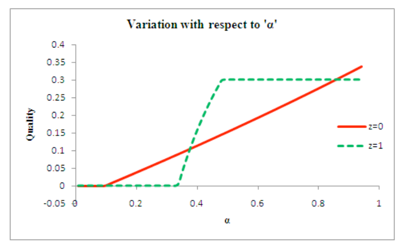
(c) Hacker's Effort ($M = 0.35$, $L = 0.05$)



(d) Patch Quality ($M = 0.05$, $L = 0.01$)



(e) Patch Quality ($M = 0.25$, $L = 0.01$)



(f) Patch Quality ($M = 0.35$, $L = 0.05$)

Figure 5: Optimal Hacker's Effort and Patch Quality - Effect of Antipiracy Effort(α).

Next, we consider Figure 5, which depicts the variation of B^* with respect to α. As α increases, the number of pirated copies available for exploit decreases for both cases of patch distribution policy and, so, the hacker's effort, B^*, also decreases for both settings of z. The key difference is in the pace at which B^* decreases – the decrease is faster when $z = 1$. The difference in the rate of decrease can be attributed to the vendor's incentive to preserve the hacker when $z = 0$ for its countervailing effect on the consumers.

Lastly, consider the variation of x^* in Figure 5. When $\alpha = 0$, there is no differentiation for $z = 1$ and $x_1^* = 0$; but x_0^* may not be zero because the vendor takes advantage of the countervailing effect to differentiate legal software from its pirated counterpart. For small enough α values, the same trade-offs occur leading to a better patch quality when $z = 0$. The rate at which patch quality increases with an increase in α is faster when patches are available freely relative to when they are restricted in distribution. This behavior can also be attributed to the vendor's incentive to drive the hacker out quickly when only the adverse effect is present. The following remark summarizes the aforementioned observations:

Numerical Observation 16. *1. For low values of α, the vendor profit and the price are higher for $z = 0$ compared to that for $z = 1$.*

2. The hacker is driven out (i.e., $B^* = 0$) at a lower value of α when $z = 1$ than when $z = 0$.

3. The rate at which the patch quality increases is higher when $z = 1$ than when $z = 0$.

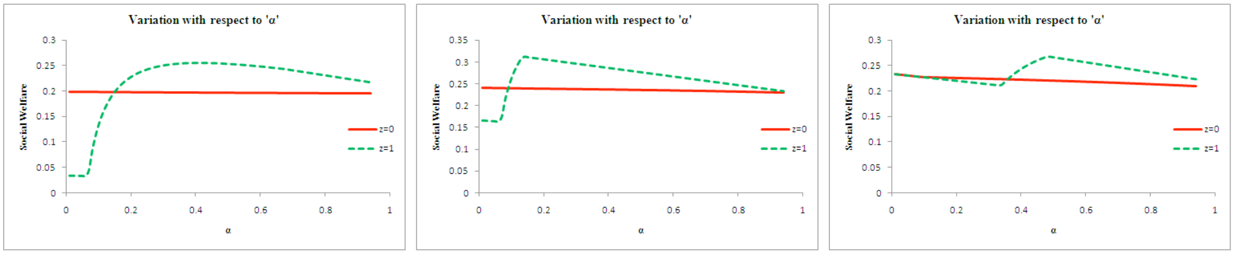
5.2 Social Welfare

This section compares the social welfare generated. Defining social welfare in our context requires care because it must be decided whether the welfare of the pirate and/or the hacker should be included in social welfare. Building on Trumbull (1990), we choose to exclude hacker's welfare but include the net benefits of all users and the vendor in our definition of social welfare. The payments from the users to the vendor are transfers, which cancel out in our calculations. Accordingly, the social welfare is:

$$\mathrm{SW} = \begin{cases} \text{Indeterminate} & \text{if } \alpha = 0 \text{ and } \begin{cases} z = 1, \text{ or} \\ z = 0 \text{ and } (x^* \text{ or } B^*) = 0 \end{cases} \\ (1 - B^*(1 - x^*\ z))\ (1-\alpha) \displaystyle\int_0^{\Theta^*} \theta^2 \ \mathrm{d}\theta & \\ +(1 - B^*(1-x^*))\ \displaystyle\int_{\Theta^*}^{1} \theta^2 \ \mathrm{d}\theta + L \log(1 - x^*) & \text{otherwise} \end{cases}$$

The first case deals with scenarios when there is no differentiation between the legal and the pirated versions. Also, as mentioned in Section 3, we do not consider the strategic role of the social planner and, hence, we do not include the cost of α in the social welfare expression. In the following, we primarily focus on studying how the social welfare varies with α.

Setting $z = 0$ and $z = 1$ in the social welfare expression we obtain SW_0 and SW_1. Figure 6 shows the variation of social welfare with α under both cases. Consider first $z = 1$, in conjunction with the corresponding variations in B^* and x^* in Figure 5. When α is so small that $x_1^* = 0$ or x_1^* is large enough that $B_1^* = 0$, the social welfare decreases as α increases because pirates lose the surplus



(a) Social Welfare ($M = 0.05$, $L = 0.01$)

(b) Social Welfare ($M = 0.25$, $L = 0.01$)

(c) Social Welfare ($M = 0.35$, $L = 0.05$)

Figure 6: Social Welfare.

due to the $(1-\alpha)$ term. When $x_1^* \neq 0$ and $B_1^* \neq 0$, there exists a range of α values where social welfare increases with α. This may be explained as follows. As anti-piracy efforts increase, fewer consumers pirate the software, which reduces the incentive of the hacker to exert effort, increasing the utility for both legal and pirate users. Beyond a certain threshold, the intense antipiracy effort decreases the surplus of the pirates drastically and, consequently, the social welfare decreases even when $B_1^* \neq 0$. (e.g., Figures 6(b) and 6(c)).

The social welfare under $z = 0$ has a more monotonic relationship with α. As noted when studying the equilibrium properties, the patch quality monotonically increases with α. This improves the surplus of the legal users. However, the surplus of the pirates decreases due to increasing α. Taken together, the overall social welfare decreases monotonically.

Interesting insights can be derived by comparing the two cases. When the antipiracy effort is low, it may not be possible for the vendor to sustain presence if the patches are distributed freely, *i.e.*, $z = 1$. Clearly, in this case, social welfare is higher when $z = 0$. As α increases, although the hacker effort aids the vendor via the countervailing effect, it also diminishes the social welfare of the society. Hence, for moderate α values, the welfare from $z = 1$ is better. This may play an important role in policy considerations.

Numerical Observation 17. 1. *The threshold α value at which the vendor prefers to release patches freely (i.e., his profits are higher with $z = 1$) is larger than the value of α when the social planner prefers the vendor to do so.*

2. *When there is little antipiracy effort, strategically exploiting hacking activity by restricting*

access to the patch to only legal users can be social welfare improving.

3. *Maximum social welfare is obtained when a moderate level of antipiracy effort is exerted and the vendor is required to release the patch freely.*

The first point in the above observation leads us to believe that the vendor does not necessarily have an incentive to freely release patches even though doing so maximizes social welfare. Instead, a government regulation may be required to make this happen. However, as the second point argues, such a regulation has to be complemented with an appropriate level of antipiracy effort. If sufficient antipiracy effort is not exerted, the society may again be worse off. This is because markets that fail in the presence of the regulation may sustain in an unregulated environment where the vendor is allowed to adopt a restricted patch distribution policy. Thus, our observations point to the need for complementing regulations with an appropriate level of antipiracy effort.

6 Discussions and Conclusion

Software vendors are taking an active role in thwarting piracy by restricting access to patches. This paper studies the implications of the vendor's decision to restrict patch distribution. Specifically, our interest is in analyzing how the vendor can exploit the strategic role of a hacker in order to maximize his profits. We employ a relatively simple model that has a reasonable level of fidelity. Nevertheless, analytical closed-form solutions describing the equilibrium strategy of the players in this game-theoretic model do not seem amenable. Despite this shortcoming, we demonstrate various results analytically and complement them with a thorough numerical analysis.

We identify two main effects of the hacker's activity on the vendor profit. The adverse effect, which occurs independent of the vendor's decision to restrict the patch, decreases the welfare of the legal users that the vendor can extract. In contrast, the countervailing effect differentiates the legal and pirated versions of the software when the patches are restricted and is therefore liked by the vendor. Whether the adverse or the countervailing effect dominates depends respectively on whether the patch quality is lower or higher than the governmental antipiracy effort. We demonstrate the interesting interplay between antipiracy efforts, patch distribution policies, and patch quality decisions on the vendor profits.

Our analysis finds that the countervailing effect has many interesting implications. We demonstrate that the often-cited motivation for restricting patch, *i.e.*, to increase market-share, is not necessarily valid when the countervailing effect is dominant. In such a case, we found that the vendor may decrease the market-share, exert more effort in providing good quality patches, and charge a higher price to generate more profit. Next, we present some policy insights arising from our analysis and also compare them against prior works.

The countervailing effect helps sustain the market in some cases. If the governmental antipiracy effort is low, the countervailing effect provides the only vertical differentiation that the vendor can use to serve the market. Stated differently, the presence of the hacker is beneficial to the market as a whole when the cost of hacking is low and the government wishes to invest little resources.

There are also differences between when the vendor and the social planner prefer to make the patch available to everyone. The government prefers that the patch be made available freely for moderate levels of antipiracy effort. The vendor only does so for a higher level of antipiracy effort. So, to achieve the best social welfare outcomes, the government has to choose an appropriate level of antipiracy effort and also require the vendor to release the patch to all users.

To the best of our knowledge, related prior works in the literature have not considered the countervailing effect. As we demonstrated in Proposition 15, if the hacker action is not endogenized, the vendor always prefers the restricted patch distribution policy. Instead, if hacker action is endogenized the vendor may have sufficient incentive to release patches freely when the government exerts sufficient antipiracy effort (see Proposition 14). Further, if hacker action is not endogenized, the vendor always targets $\frac{1}{3}$rd of the market and we do not see the situation where the vendor improves patch quality and increases price to target a smaller segment of consumers. In August and Tunca (2008) and Lahiri (2012), features such as network externality generate the differentiation between the restricted patch and no restricted patch scenarios. As a consequence, some of the comparative statics results observed are different compared to our model. For example, August and Tunca (2008) find cases when the intense antipiracy enforcement should be complemented with the decision to restrict the patch distribution. However, that issue does not arise in our context.

The next obvious question is: how robust are our results to relaxing various assumptions, including the uniform distribution of consumers, the functional form of consumer and the hacker utilities, etc.? To address this question, we provide features of the model that are critical to establishing the results. If the probability terms used in Equations 1 and 2 are retained the same but the functional form of the consumer utility/demand is made more general, the indifferent consumer type will continue to include a term $(x - \alpha)$. This is the term that drives the main results in our paper about the countervailing versus the adverse effect trade-off. As one may realize, the term will continue to play a similar qualitative role even in a generic setting although the second order effects from the generic functions may affect the specific values. If one were to pay careful attention to the proofs, an important feature of the model is the single-crossing property of Lemma 4, which helps establish the uniqueness of B and Θ. Another important feature of the model for analytical results is that the first partial derivative of vendor's profits is a rational function that enables us to use semialgebraic geometry tools for the analysis. Once we discovered the results of Theorems 10 and 11, we found that they have quite intuitive explanations that primarily depend on how β interacts with $(x - \alpha)$ (see discussion after Theorem 11). We also performed some numerical computations with other monomial utility functions for consumer types and found that the results we derived extend to those settings as well. For these reasons, we conjecture reasonably confidently that our results should be valid more generally.

Although our results provide interesting insights regarding the implications of restricted patch distribution, our analysis is not without limitations. The key limitation of our study is that some of the insights were derived through numerical computations, which were performed by creating a grid with a granularity of 0.01 for the problem parameters. In Example 9, we provided some justification for our approach. Second, in our model we assume that if the patch is available to the user, the user will patch the system. However, this may not be the case. Thirdly, implementing the restricted patch distribution policy need not be costless. Incorporating these features, performing analytical investigations with more general cost and utility functional forms would be interesting avenues for further research. Further, it would be interesting to investigate if the results found in our analysis can be validated empirically.

A Proofs

A.1 Proof of Lemma 1

Since $\pi_h(0) = 0$, it follows that $\sup_\beta \pi_h(\beta) = \sup_\beta\{\pi_h(\beta) \mid \pi_h(\beta) \geq 0\}$. However,

$$\beta(1 - x\ z)\int_0^{\bar{\theta}} \theta\ \mathrm{d}\theta + \beta(1 - x)\int_{\bar{\theta}}^1 \theta\ \mathrm{d}\theta \leq \beta(1 - xz)\int_0^1 \theta\ \mathrm{d}\theta \leq \frac{\beta(1 - xz)}{2} \leq \frac{\beta}{2}$$

Therefore, any β such that $\pi_h(\beta) \geq 0$ satisfies $C(\beta) \leq \frac{\beta}{2} \leq \frac{1}{2}$. In other words, for a given L, there exists an $\epsilon > 0$ such that $\beta \leq 1 - \epsilon$. For a given $\bar{\theta}$, $\pi_h(\beta)$ is continuous over $[0, 1 - \epsilon]$. Therefore, by Weierstrass theorem, $\sup_\beta \pi_h(\beta)$ is attained at some $\beta \leq 1 - \epsilon$.

A.2 Proof of Lemma 2

First note that because $\kappa(\alpha, \beta, x, z) \leq 1$, we have already argued that any equilibrium can be represented with a software price that is less than or equal to one. Define $S = [0, 1) \times [0, 1]$ and $\Pi_v = \sup_{x,p}\{\pi(x, p) \mid (x, p) \in S\}$. Since $(0, 0) \in S$, it follows that $\Pi_v \geq 0$. Let $S' = \{(x, p) \mid \pi(x, p) \geq 0\}$. It follows that $\Pi_v = \sup_{x,p}\{\pi(x, p) \mid (x, p) \in S'\}$. We now show that S' is compact. If $(x', p') \in S'$, then $L \log(1 - x') \geq -1$. Let $S'' = \left[0, 1 - \exp\left(-\frac{1}{L}\right)\right] \times [0, 1]$. Then, $S' \subseteq S''$. It is easy to verify that $\pi(x, p)$ is continuous over S'' and therefore $S' = S'' \cap \{(x, p) \mid \pi(x, p) \geq 0\}$ is compact. Using Weierstrass Theorem, it follows that the supremum in the definition of Π_v is attained. Let the optimal solution to the vendor's problem be (x^*, p^*). We have shown in the discussion before the lemma, that p^* may be restricted to lie in $[0, \kappa(\alpha, \beta, x^*, z)]$. We now argue that $p^* \in \big(0, \kappa(\alpha, \beta, x^*, z)\big)$. We proceed by contradiction. First note that if p is set at any of the boundary points the vendor revenue is 0 and therefore $\pi(x^*, p) = 0$. We show that there exists another solution (x', p') such that $\pi(x', p') > 0$. Let $x' = 0$ and $p' = \kappa(\alpha, \beta, x^*, z)/2$. Since $\kappa(\alpha, \beta, x^*, z) > 0$, $p' > 0$ and it can be easily verified that $\bar{\theta} < 1$. Therefore, $\pi(x', p') > 0$. Since $p^* \in \big(0, \kappa(\alpha, \beta, x^*, z)\big)$ it follows that $\bar{\theta} \in (0, 1)$. □

A.3 Proof of Lemma 4

Consider u' such that $h(u') - \max_{i\in I} g_i(u') \leq 0$ and $u'' > u'$. Let $\hat{i}$ be such that $g_{\hat{i}}(u') = \max_{i\in I} g_i(u')$. Then, $h(u'') - \max_{i\in I} g_i(u'') \leq h(u'') - g_{\hat{i}}(u'') < 0$, where the second inequality follows since $h(u)$ single crosses $g_{\hat{i}}(u)$ from above by definition. $\square$

A.4 Proof of Lemma 5

By Lemma 1, let $\beta^* = \arg\max_\beta h_\pi(\beta)$. Now, by Fermat's theorem, $\beta^* \in \left\{0, \hat{\beta} = 1 - \frac{2M}{1-x\left(1-(1-z)\bar{\theta}^2\right)}\right\}$. Observe that $\pi_h(\hat{\beta}) > \pi_h(0) = 0$. If $\hat{\beta} > 0$ then $\beta^* = \hat{\beta}$ else $\beta^* = 0$. Therefore, the equilibrium solution $(\bar{\theta}^*, \beta^*)$ is obtained by solving (3) and (5) simultaneously for $\bar{\theta}$ and β. Since $M > 0$ and $1 - x(1-(1-z)\bar{\theta}^2) \geq 0$, it follows that if the equilibrium is attained at $(\bar{\theta}^*, \beta^*)$ then $\beta^* < 1$, which was also shown in Lemma 1. Finally, if $M \geq \frac{1}{2}$, then $\hat{\beta} < 0$. Therefore, $\beta^* = 0$.

We now prove that there is a unique solution to (3) and (5). When $z = 1$ or $x = \alpha$, it can be trivially shown that (3) and (5) intersect at a unique point given x, α, p, and M. This is because in the former case (5) is independent of $\bar{\theta}$ and if $z = 0$ and $x = \alpha$ then (3) does not depend on β. Therefore, we assume for the remaining proof of uniqueness that $x \neq \alpha$ and $z = 0$. Let $r(\beta) = \sqrt{\frac{p}{\alpha(1-\beta)+\beta x}}$. If $p = 0$, it is easy to show that there is a unique solution to (3) and (5). Therefore, we may assume $p > 0$ and consequently $\bar{\theta}^* > 0$. Since $x \neq \alpha$, it follows that $r(\beta)$ is strictly monotone in β and therefore invertible. By direct calculation, $r^{-1}(\theta) = -\frac{p}{\theta^2(\alpha-x)} + \frac{\alpha}{\alpha-x}$. Since $r(\beta)$ is strictly monotone, $r^{-1}(\theta)$ is strictly monotone in θ. Let $s(\theta) = 1 - \frac{2M}{1-x(1-\theta^2)}$ and define $w(\theta) = r^{-1}(\theta) - s(\theta)$. Then:

$$w(\theta) = r^{-1}(\theta) - s(\theta) = \frac{-p + px - px\theta^2 + x\theta^2 - x^2\theta^2 + \theta^4 x^2 + 2M\theta^2\alpha - 2M\theta^2 x}{\theta^2(\alpha - x)(1 - x + x\theta^2)}$$

We show that there do not exist $\theta', \theta'' \in (0,1]$, $\theta' \neq \theta''$, such that $w(\theta') = w(\theta'') = 0$. It is easy to verify that the denominator of $w(\theta)$ is non-zero when $\theta \in (0,1]$. Replace $u = \theta^2$ and observe that the numerator is negative when $u = 0$. Further, as u increases, the numerator becomes positive because either $x > 0$ in which case the coefficient of u^2 is positive or $x = 0$ in which case the coefficient of u^2 is zero and the coefficient of u is $2M\alpha$, which in turn is strictly positive. Since the numerator is quadratic in u, it has at most one root in $(0,1]$. Therefore, $w(\theta)$ has at most one root in $(0,1]$. Since $r^{-1}(\theta)$ and $s(\theta)$ are continuous, it follows that, for $\theta \in (0,1]$, $r^{-1}(\theta)$ single-crosses $s(\theta)$ either from above or below. Further, since $r^{-1}(\theta)$ is strictly monotonic, it is easy to verify that if $r^{-1}(\theta)$ crosses $s(\theta)$ from above (resp. below) then it also crosses the identically zero function from above (resp. below). The former occurs when $\alpha < x$ and the latter when $\alpha > x$. Then, it follows from Lemma 4 that $r^{-1}(\theta)$ crosses $\max\{0, s(\theta)\}$ either from above or below.

Let $S_1 = \{(\min\{r(\beta), 1\}, \beta) \mid 0 \leq \beta \leq 1\}$ and $S_2 = \{(\theta, \max\{0, s(\theta)\}) \mid 0 \leq \theta \leq 1\}$. We now show that $S_1 \cap S_2$ is a singleton. Assume that $\alpha < x$. Then, $r^{-1}(\theta)$ single-crosses $\max\{0, s(\theta)\}$ from above and $r^{-1}(\theta)$ is strictly monotonically decreasing. Now consider points $(\theta, \beta) \in S_1$ such that $r(\beta) \geq 1$. Clearly, $\theta = 1$. Also, observe that as θ approaches ∞, $r^{-1}(\theta)$ approaches $\frac{\alpha}{\alpha - x}$, which in turn is negative. It follows as a result that all points of the form $(1, \beta)$, where $0 \leq \beta \leq \min\{r^{-1}(1), 1\}$, belong to S_1. Observe that there is only one element in S_2 such that $\theta = 1$. Then, S_1 intersects with S_2 at such a point if and only if $\min\{r^{-1}(1), 1\} \geq \max\{0, s(1)\}$. Since $\max\{0, s(1)\} < 1$, the condition reduces to $r^{-1}(1) \geq \max\{0, s(1)\}$. However, as shown above, $r^{-1}(\theta)$ single-crosses $\max\{0, s(\theta)\}$ from above. This implies that $r^{-1}(\theta) > \max\{0, s(\theta)\}$ for $\theta < 1$. In other words, if $\alpha < x$, $S_1 \cap S_2$ is a singleton. Now, consider the case when $\alpha > x$. In this case, $r^{-1}(\theta)$ crosses $\max\{0, s(\theta)\}$ from below and is monotonically increasing. Further, as θ approaches ∞, $r^{-1}(\theta)$ approaches $\frac{\alpha}{\alpha - x}$, which is at least one. Therefore, points of the form $(1, \beta)$ belong to S_1 if and only if $1 \geq \beta \geq \max\{r^{-1}(1), 0\}$. Such a point belongs to S_2 if and only if $\max\{0, s(1)\} \geq \max\{r^{-1}(1), 0\}$ which is equivalent to $r^{-1}(1) \leq \max\{0, s(1)\}$. However, since $r^{-1}(1)$ crosses $\max\{0, s(1)\}$ from below, this implies that $r^{-1}(\theta) < \max\{0, s(\theta)\}$ for $\theta < 1$. In other words, if $\alpha > x$, $S_1 \cap S_2$ is a singleton. Therefore, the uniqueness of $(\bar{\theta}^*, \beta^*)$ is proven. □

A.5 Proof of Proposition 6

We consider the decision variables p and x separately. First note that by Lemma 2, we may assume that $p < \kappa(\alpha, \beta, x, 0)$ and that, given x, B is independent of p. Therefore, we solve for p first: $p^* = \frac{4\alpha \min\{2M+x,1\}}{9}$. The first order optimality conditions fix $\Theta^* = \frac{2}{3}$. Then, by Lemma 2, the vendor's decision problem for deciding patch quality reduces to $\max\left\{\pi_v(x) \mid x \in \left[0, 1 - \exp\left(-\frac{1}{L}\right)\right]\right\}$, where $\pi_v(x) = \frac{4\alpha \min\{2M+x,1\}}{27} + L \log(1 - x)$. For $x > (1-2M)$, it is easy to check that $\pi_v(x) < \pi_v(1-2M)$ because the first term is constant and the second term is strictly decreasing. Consequently, we may additionally restrict x to $[0, 1-2M]$. Observe that the $\pi_v(x)$ is strictly concave in x and, therefore, the optimal patch quality is unique. Further, express $\pi_v(x)$ as $\frac{4\alpha(2M+x)}{27} + L \log(1 - x)$ and observe that it is differentiable w.r.t. x. It is easy to verify that $\pi_v(0) > 0 \geq \pi_v\left(1 - \exp\left(-\frac{1}{L}\right)\right)$. So, by Fermat's theorem, $x^* \in \left\{0, 1 - \frac{27L}{4\alpha}, 1 - 2M\right\}$. (Note that $1 - 2M > 0$ since $M < \frac{1}{2}$.) The proposition lists the various cases that occur depending on which value of x maximizes vendor's profit. $\square$

A.6 Proof for Lemma 8

We first show that $x^* \leq q(\Theta) = \frac{1-2M}{1-\Theta^{*2}}$. Assume for deriving a contradiction that $x^* > q(\Theta^*)$ and the corresponding software price is p^*. Then, it follows from (7) that $B^* = 0$. Now consider the situation where the vendor chooses the patch quality x'. Observe from (7) that the equilibrium market share is independent of x when hacker is not exerting an effort. Therefore, the corresponding equilibrium solution to the second stage problem is $(\beta^*, \bar{\theta}^*) = (0, \Theta^*)$. It follows from the definition of $\pi'_v(x, \Theta)$ (see (8)) that $\pi_v(x', \Theta^*) > \pi_v(x^*, \Theta^*)$. Therefore, (x^*, Θ^*) does not maximize $\pi_v(x, \Theta)$ contradicting our assumption.

Now, we prove the last statement of the result. If $B^* = 0$, then it follows from the above argument that $x^* \leq q(\Theta^*)$. However, $B^* = 0$ implies that $x^* \geq q(\Theta^*)$. Therefore, $x^* = q(\Theta^*)$. It is easy to check that $(1-\Theta)\Theta^2\alpha$ is a decreasing function for $\Theta \in \left[\frac{2}{3}, \infty\right]$. Further, since $M < \frac{1}{2}$, $L\log(1-x)$ is a decreasing function of Θ if $x = q(\Theta)$. Therefore, $\Theta^* \leq \frac{2}{3}$. $\square$

A.7 Proof for Theorem 10

Proof for Point #1: Recall that $\Theta^* \in (0, 1)$. By simple calculation,

$$\frac{\partial^2 \pi_v(x,\Theta)}{\partial x^2} = -\frac{4M(1-\Theta)^2\Theta^2(1+\Theta)(1-\alpha(1-\Theta^2))}{(1-(1-\Theta^2)x)} - \frac{L}{(1-x)} < 0 \tag{11}$$

$$\frac{\partial^2 \pi_v(x,\Theta)}{\partial x \partial M} = -2(1-\Theta)\Theta^2\frac{(1-\alpha(1-\Theta^2))}{(1-x(1-\Theta^2))^2} < 0. \tag{12}$$

It follows from (11) that $\pi_v(x, \Theta)$ is strictly concave for a fixed Θ. Therefore, $x^*(\Theta)$ is unique. By (12), $\frac{\partial^2 \pi_v(x,\Theta)}{\partial x \partial M} < 0$. Therefore, as M increases $x^*(\Theta)$ strictly decreases.

Now, we show that when $L > \frac{4}{27}$, $x^*(\Theta) = 0$. Since $x^*(\Theta)$ is strictly decreasing in M, the result holds if and only if it holds for $M = 0$. Therefore, we assume that $M = 0$. By Fermat's theorem, $x^*(\Theta) = \max\left\{0, 1 - \frac{L}{(1-\Theta)\Theta^2}\right\}$. Let $x' = 1 - \exp\left(-\frac{1}{L}\right)$ and $x''(\Theta) = \frac{1}{1-\Theta^2}$. Observe that $x^*(\Theta) \neq x'$ because $\pi_v(x', \Theta) < \pi_v(0, \Theta)$ and $x^*(\Theta) \neq x''(\Theta)$ because $x''(\Theta) \geq 1$. Since $(1-\Theta)\Theta^2 \leq \frac{4}{27}$, it follows that $x^*(\Theta) \leq \max\left\{0, 1 - \frac{27L}{4}\right\}$. Therefore, if $L \geq \frac{4}{27}$ it follows that $x^*(\Theta) = 0$ for all Θ and, therefore, $x^* = 0$. $\square$

Proof for Point #2: We consider $\pi_v(x, \Theta)$ and its variation over $\Theta \in (0, 1)$. First, we define

$t = \sqrt{\max\left\{\frac{x-1+2M}{x}, 0\right\}}$. It can be checked that $\Theta' \geq t$ if $t < \frac{2}{3}$ because $\pi_v(x,t) > \pi_v(x,\Theta)$ for all $\Theta < t$. Further, if $t > \frac{2}{3}$ then $\pi_v\left(x, \frac{2}{3}\right) > \pi_v(x,\Theta)$ for all $\Theta < t$. Moreover, for $\Theta \geq t$, $\pi_v(x,\Theta)$ reduces to:

$$\pi_v(x,\Theta) \quad = \quad \underbrace{(1-\Theta)\Theta^2}_{q(\Theta)} \underbrace{\left((\alpha - x)\frac{2M}{(1-(1-\Theta^2)x)} + x\right)}_{h(\alpha,x,M,\Theta)} + L\log(1-x).$$

It is easy to check that $q'(\Theta) = \frac{\mathrm{d}q(\Theta)}{\mathrm{d}\Theta} = -3\Theta^2 + 2\Theta$, which is positive (resp. negative) for $\Theta \in \left(0, \frac{2}{3}\right)$ (resp,. $\Theta \in \left(\frac{2}{3}, 1\right)$) and is zero when $\Theta = \frac{2}{3}$. Further, $q(\Theta) > 0$ for $\Theta \in (0,1)$. We now show that $h(\alpha,x,M,\Theta) > 0$. Observe that $\frac{2M}{1-x(1-\Theta^2)} \leq 1$. Therefore, if $x \geq \alpha$, $h(\alpha,x,M,\Theta) \geq \alpha > 0$. If $x < \alpha$ then $\frac{2M}{1-x(1-\Theta^2)} > 0$ and consequently $h(\alpha,x,M,\Theta) > 0$. Now, observe that $h'(\alpha,x,M,\Theta) = \frac{\partial h(\alpha,x,M,\Theta)}{\partial\Theta} = \frac{-4M(\alpha-x)\Theta x}{(1-(1-\Theta^2)x)^2}$. Then, it is easy to see that if $\alpha = x$ or $x = 0$ then $h'(\alpha,x,M,\Theta) = 0$. Otherwise, $h'(\alpha,x,M,\Theta) < 0$ if $\alpha > x$ and $h'(\alpha,x,M,\Theta) > 0$ if $0 < x < \alpha$.

Let $\pi_v'(x,\Theta) = \frac{\partial \pi_v(x,\Theta)}{\partial\Theta}$. Then, $\pi_v'(x,\Theta) = q'(\Theta)h(\alpha,x,M,\Theta) + q(\Theta)h'(\alpha,x,M,\Theta)$. It is easy to check that if $x = 0$ then $t = 0$ and therefore $\Theta' = \frac{2}{3}$. If $x = \alpha$, it can be checked that the two cases in the definition of $\pi_v(x,\Theta)$ are identical and therefore $\Theta' = \frac{2}{3}$. If $0 < x < \alpha$, then for all $\Theta \geq \max\left\{t, \frac{2}{3}\right\}$, it can be easily seen that $q'(\Theta)h(\alpha,x,M,\Theta) \leq 0$ and $q(\Theta)h'(\alpha,x,M,\Theta) < 0$. Therefore, $\pi_v'(x,\Theta) < 0$. In other words, if $t < \frac{2}{3}$, $\Theta' < \frac{2}{3}$. On the other hand, if $t > \frac{2}{3}$ then $\pi_v\left(x, \frac{2}{3}\right) > \pi_v(x,t) > \pi_v(x,\Theta)$ for all $\Theta > t$. Therefore, $\Theta' = \frac{2}{3}$. Now, consider $x > \alpha$. In this case, for all $t \leq \Theta \leq \frac{2}{3}$ it follows that $q'(\Theta)h(\alpha,x,M,\Theta) \leq 0$ and $q(\Theta)h'(\alpha,x,M,\Theta) < 0$. Therefore, $\pi_v'(x,\Theta) < 0$. It follows that $\Theta^*(x) \in \left(\frac{2}{3}, 1\right)$. □

Proof for Point #3: The proof has three parts. The first part establishes that there exist $\{\alpha, L, M\}$ such that $(x^*, \Theta^*) = \left(\alpha, \frac{2}{3}\right)$ is the unique optimal solution to the vendor's problem. The second part of the proof shows that when problem parameters are perturbed slightly $x^* \in (0,1)$. The last part deals with the impact of a perturbation of M on x^*.

First part: Fix α, L, and M. Define $\acute{\pi}_v(x,\Theta) = (1-\Theta)\Theta^2((\alpha - x)\frac{2M}{1-(1-\Theta^2)\alpha} + x) + L\log(1-x)$. Observe that $\acute{\pi}_v(x,\Theta)$ is obtained by replacing the x in the denominator of the first term of $h(\alpha,x,M,\Theta)$ by α. For $0 < \alpha < 1$, the first term in the definition of $\acute{\pi}_v(x,\Theta)$ is bounded for all $(x,\Theta) \in [0,1]^2$. Therefore, it is easy to verify that there exists a point that maximizes $\acute{\pi}_v(x,\Theta)$ over $[0,1]^2$. Let $S^\Pi = \arg\max\{\acute{\pi}_v(x,\Theta) \mid (x,\Theta) \in [0,1]^2\}$. It can be easily verified that $\acute{\pi}_v(x,0) \leq 0$,

$\acute{\pi}_v(x,1) \leq 0$ and $\acute{\pi}_v(0,0.5) = \frac{\alpha}{4-3\alpha} > 0$. Therefore, if $(x,\Theta) \in S^{\Pi}$ then $0 < \Theta < 1$. Let S^{π} denote the set of optimal solutions to (9).

Lemma 18. *If $\Theta \in (0,1)$ and $x \leq \frac{1-2M}{1-\Theta^2}$ then $\acute{\pi}_v(x,\Theta) \geq \pi_v(x,\Theta)$ where the inequality is strict if $x \neq \alpha$. If $(\alpha,\Theta') \in S^{\Pi}$ then $S^{\pi} = (\alpha, \frac{2}{3})$.*

Proof. Observe that $1-(1-\Theta^2)\alpha < 1-(1-\Theta^2)x$ if and only if $\alpha > x$. Then, it follows that $(\alpha - x)\frac{2M}{1-(1-\Theta^2)\alpha} + x \geq h(\alpha, x, M, \Theta)$ where strict inequality holds if $\alpha \neq x$. Therefore, the first statement is proven. Now, let $(x^*,\Theta^*) \in S^{\pi}$ and $(\alpha,\Theta') \in S^{\Pi}$. Then:

$$\pi_v(\alpha,\Theta') = \acute{\pi}_v\left(\alpha,\Theta'\right) \geq \acute{\pi}_v(x^*,\Theta^*) \geq \pi_v(x^*,\Theta^*) \geq \pi_v(\alpha,\Theta') \tag{13}$$

where the first equality is because $\acute{\pi}_v(x,\Theta) = \pi_v(x,\Theta)$ whenever $x=\alpha$, the first inequality because $(\alpha,\Theta') \in S^{\Pi}$, the second inequality by the first statement of the lemma and because $x^* \leq \frac{1-2M}{1-\Theta^2}$ by Lemma 8, and the third inequality because $(x^*,\Theta^*) \in S^{\pi}$. Therefore, equality holds throughout and $(\alpha,\Theta') \in S^{\pi}$. Further, the equality cannot hold if $x^* \neq \alpha$ because the second inequality in (13) is strict in this case. Therefore, the result follows by Point #2 above. $\square$

Lemma 19. *$\exists\{\alpha, L, M\}$ such that $\alpha < \frac{9}{5}(1-2M)$ and $(\alpha, \frac{2}{3}) = S^{\Pi} = S^{\pi}$.*

Proof. Let $\alpha = \frac{1}{2}$, $L = \frac{1}{18}$ and $M = \frac{13}{144}$. It can be verified that $\alpha < \frac{9}{5}(1-2M)$. Our primary goal is to show that $S^{\Pi} = \left(\alpha, \frac{2}{3}\right)$. Then, it will follow from Lemma 18 that $S^{\pi} = \left(\alpha, \frac{2}{3}\right)$. Instead of showing that $S^{\Pi} = \left(\alpha, \frac{2}{3}\right)$, we will prove a slightly more general statement. Let $S_1 = [0, 0.32) \cup (0.92, 1]$, $S_2 = [0.32, 0.92]$, $X_1 = [0,1]$, and $X_2 = [-1,1]$. We will show that $\left(\alpha, \frac{2}{3}\right)$ is the maximizer of $\acute{\pi}_v(x,\Theta)$ over $R = X_1 \times S_1 \cup X_2 \times S_2$. First, observe that we have already shown that there is a maximizer for $\acute{\pi}_v(x,\Theta)$ over $[0,1]^2$. It is easy to check that there exists a maximizer of $\acute{\pi}_v(x,\Theta)$ over R because $[-1,0] \times [0.32, 0.92]$ is a compact set and $\acute{\pi}_v(x,\Theta)$ is continuous over the region. Given Θ, $\acute{\pi}_v(x,\Theta)$ is strictly concave and the unconstrained maximizer can be computed as follows:

$$\hat{x}(\Theta) = \frac{-21\Theta^2 - 36\Theta^4 + 23\Theta^3 + 36\Theta^5 + 2}{\Theta^2(-1+\Theta)(36\Theta^2+23)}. \tag{14}$$

Consider $(x,\Theta) \in X_1 \times S_1$. We have already argued that any points with $\Theta \in \{0,1\}$ do not attain the maximum. Now, let $\Theta \in S_1 \backslash \{0,1\}$. It follows that $x^*(\Theta) = \max\{\hat{x}(\Theta), 0\}$. Then, it can be verified using Sturm's theorem that $\hat{x}(\Theta) < 0$ and therefore $x^*(\Theta) = 0$. Also, $\frac{\mathrm{d}\acute{\pi}_v(0,\Theta)}{\mathrm{d}\Theta} = -\frac{13}{72}\Theta\left(\frac{2-3\Theta-\Theta^3}{(1+\Theta^2)^2}\right)$. An application of Sturm's theorem shows that $2 - 3\Theta - \Theta^3 \neq 0$ for $\Theta \in S_1$. Therefore, none of the points in $X_1 \times S_1$ is a maximizer.

Now, we consider the region $X_2 \times S_2$. For a given $\Theta \in S_2$, it follows that $x^*(\Theta) = \max\{\hat{x}(\Theta), -1\}$. However, an application of Sturm's theorem shows that, when $\Theta \in S_2$, the numerator in (14) is bounded from above by 1 and the denominator is bounded from above by -1. Therefore, $\hat{x}(\Theta) \geq -1$. It follows that $x^*(\Theta) = \hat{x}(\Theta)$. Let $\pi(\Theta) = \acute{\pi}_v(\hat{x}(\Theta), \Theta)$. Then, it can be verified that $\frac{\mathrm{d}\pi(\Theta)}{\mathrm{d}\Theta}$ vanishes if and only if $\Theta = \frac{2}{3}$ or $p_2(\Theta) = 2592(\Theta^9 - \Theta^8) + 6684\Theta^7 - 6644\Theta^6 + 5371\Theta^5 - 4965\Theta^4 + 1383\Theta^3 - 977\Theta^2 + 92$ vanishes. By Sturm's theorem, $p_2(\Theta)$ does not vanish at any $\Theta \in S_2$. Further, $\frac{\mathrm{d}\pi(\Theta)}{\mathrm{d}\Theta}$ is positive when $\Theta = 0.32$ and negative when $\Theta = 0.92$. Therefore, we may restrict attention to $\Theta = \frac{2}{3}$. Then, it follows from (14) that $\left(\alpha, \frac{2}{3}\right) = S^{\Pi}$. $\square$

Second Part: Let $\{\alpha, L, \hat{M}\}$ be such that $S^{\pi} = \left(\alpha, \frac{2}{3}\right)$ and $\alpha < \frac{9}{5}(1 - 2M)$. Then, consider a sequence $M_\gamma \to \hat{M}$ indexed by γ. For a sufficiently large γ, the feasible set is not empty. Then, by Weierstrass Theorem there exists a $(x_\gamma, \Theta_\gamma)$ that is optimal to the vendor's problem with parameters $\{\alpha, L, M_\gamma\}$. Since $(x_\gamma, \Theta_\gamma)$ belong to a compact set, by restricting to a subsequence if necessary, $(x_\gamma, \Theta_\gamma) \to (x', \Theta')$. By Berge's Maximum Theorem, (x', Θ') is an optimal solution to the vendor's problem at $M = \hat{M}$. Therefore, $(x', \Theta') = \left(\alpha, \frac{2}{3}\right)$. In other words, for any $\delta > 0$ there is a sufficiently large γ such that $\left|(x_\gamma, \Theta_\gamma) - \left(\alpha, \frac{2}{3}\right)\right| < \delta$. Since $\alpha < \frac{9}{5}(1 - 2M)$, for large enough γ, $B^* > 0$.

Third Part: Fix $\{\alpha, L, \hat{M}\}$ to be such that $S^{\pi} = \left(\alpha, \frac{2}{3}\right)$ and $\alpha < \frac{9}{5}(1 - 2M)$. Let $\pi(x, \Theta, M)$ be the vendor's profit for patch quality x, untapped market Θ, and hacker cost parameter M. Let $\epsilon > 0$ and define:

$$\Delta(x, \Theta, \epsilon) = \pi(x, \Theta, M + \epsilon) - \pi(x, \Theta, M) = \frac{(1 - \Theta)\Theta^2(\alpha - x)}{(1 - (1 - \Theta^2)x)}\epsilon. \tag{15}$$

Then, if $x \geq \alpha$, it follows that $\Delta(x, \Theta, \epsilon) \leq 0$. It follows that:

$$\pi\left(\alpha, \frac{2}{3}, \hat{M} + \epsilon\right) = \pi\left(\alpha, \frac{2}{3}, \hat{M}\right) > \pi(x, \Theta, \hat{M}) \geq \pi(x, \Theta, \hat{M} + \epsilon), \tag{16}$$

where the first equality follows from $\Delta\left(\alpha, \frac{2}{3}, \epsilon\right) = 0$, the second inequality since $S^{\pi} = \left(\alpha, \frac{2}{3}\right)$, and third inequality because $\Delta(x, \Theta, \epsilon) \le 0$. Therefore, if (x', Θ') is optimal when $M = \hat{M} + \epsilon$, either $x' < \alpha$ or $(x', \Theta') = \left(\alpha, \frac{2}{3}\right)$. Since $S^\pi = \left(\alpha, \frac{2}{3}\right)$, it follows that $\frac{\mathrm{d}\pi_v\left(x, \frac{2}{3}\right)}{\mathrm{d}x} = 0$ when $M = \hat{M}$. Then, by (12), for sufficiently small ϵ, $\frac{\mathrm{d}\pi_v\left(x, \frac{2}{3}\right)}{\mathrm{d}x} < 0$ at $M = \hat{M} + \epsilon$. Therefore, $\left(\alpha, \frac{2}{3}\right)$ is not optimal at $M = \hat{M} + \epsilon$. Similarly, for sufficiently small $\epsilon < 0$, we can show that $x^* > \alpha$ when $M = \hat{M} - \epsilon$. □

A.8 Proof for Theorem 11

By Lemma 19, there exist $\{\alpha, L, \hat{M}\}$ such that $S^\pi = \left(\alpha, \frac{2}{3}\right)$. Let $S_M = [\hat{M} - \epsilon, \hat{M}) \cup (\hat{M}, \hat{M} + \epsilon]$. Let $M' \in S_M$ and (x', Θ') be the optimal vendor strategy when $M = M'$. Then, $\pi(x', \Theta', M') > \pi\left(\alpha, \frac{2}{3}, M'\right) = \pi\left(\alpha, \frac{2}{3}, M\right)$, where the first inequality follows because $\left(\alpha, \frac{2}{3}\right)$ is not optimal when $M = M'$ and the first equality because $\pi(x, \Theta, M)$ is independent of M when $\alpha = x$. Therefore, the profit at $M = \hat{M}$ is the minimum for small enough perturbations of M.

A.9 Proof of Proposition 14

Assume that $\alpha = 1$. Then, using (3), define $\Theta^f(p, B, x) = \sqrt{\frac{p}{1 - B(1-x)}}$. Let $\pi(p, x, \Theta) = p(1 - \Theta) + L\log(1 - x)$ denote the vendor's profit. Now, we show that $\pi(p_0^*, x_0^*, \Theta_0^*) \le \pi(p_1^*, x_1^*, \Theta_1^*)$. Let $B_1^f(x)$ (resp. $B_0^f(x, \Theta_0)$) denote the hacker effort when $z = 1$ (resp. $z = 0$ and untapped market is Θ_0) and the patch quality is x. By (5), $B_1^f(x) = B_0^f(x, 0)$ and B_0^f is non-decreasing in the second argument, and strictly increasing if $x > 0$. Therefore, $B_1^f(x) \le B_0^f(x, \Theta_0^*)$. Then, it follows that $\pi(p_0^*, x_0^*, \Theta_0^*) \le \pi(p_0^*, x_0^*, \Theta^f(p_0^*, B_1^f(x_0^*), x_0^*)) \le \pi(p_1^*, x_1^*, \Theta_1^*)$. The first inequality follows because $\Theta_0^* = \Theta^f(x_0^*, B_0^f(x_0^*, \Theta_0^*), x_0^*)$, $x_0^* \le 1$, and $B_1^f(x_0^*) \le B^f(x_0^*, \Theta_0^*)$ and the second inequality holds because $(p_0^*, x_0^*, \Theta^f(p_0^*, B_1^f(x_0^*), x_0^*))$ is an admissible strategy for the vendor when $z = 1$.

First observe that $(x = 0, \Theta)$ is feasible at $z = 1$ if and only if it is also feasible when $z = 0$. To see this, observe that (5) implies that when $x = 0$, $B_1^f(x) = B_0^f(x, \Theta) = B$ and, then the untapped market is $\Theta^f(p, B, x)$.

We remark that the vendor profit does not depend on the patch distribution strategy when $x_1^* = 0$. To see this, note that $\pi(p_1^*, x_1^*, \Theta_1^*) = \pi(p_1^*, 0, \Theta_1^*) \ge \pi(p_0^*, x_0^*, \Theta_0^*) \ge \pi(p_1^*, 0, \Theta_1^*)$, where the the first inequality follows from the discussion in the previous paragraph, the second inequality is from the optimality of (x_0^*, Θ_0^*) and feasibility of $(0, \Theta_1^*)$ for $z = 0$. However, from Proposition 6, whenever $L < \frac{4}{27}$, we have $x_1^* \ne 0$. In the discussion following Theorem 10, we ruled out $L \ge \frac{4}{27}$ as being uninteresting. Therefore, we may assume that the above case does not occur.

Now, consider the case when $x_1^* > 0$. If $x_0^* = 0$ then $\pi(p_0^*, x_0^*, \Theta_0^*) = \pi(p_0^*, 0, \Theta_0^*) < \pi(p_1^*, x_1^*, \Theta_1^*)$, where the strict inequality follows because $x_1^* > 0$ and $(p_0^*, 0, \Theta_0^*)$ is feasible when $z = 1$. Now, let $x_0^* > 0$. Consider $B_0^* > 0$. By Lemma 2, $\Theta_0^* < 1$. Therefore, $B_0^* > B_1^f(x_0^*)$ and so $\Theta_0^* > \Theta^f(p_0^*, B_1^f(x_0^*), x_0^*)$. Because $p_0^* > 0$ by Lemma 2, $\pi(p_0^*, x_0^*, \Theta_0^*) < \pi(p_0^*, x_0^*, \Theta^f(p_0^*, B_1^f(x_0^*), x_0^*)) \leq \pi(p_1^*, x_1^*, \Theta_1^*)$. Now, let $B_0^* = 0$ and define $x' = x_0^*(1 - \Theta_0^{*2})$ Then, it follows that

$$\pi(p_0^*, x_0^*, \Theta_0^*) < \pi(p_0^*, x', \Theta^f(p_0^*, B_1^f(x'), x')) \leq \pi(p_1^*, x_1^*, \Theta_1^*).$$

Here, the first inequality follows because $x' < x_0^*$, $B_1^f(x') = B^f(x_0^*, \Theta_0^*) = B_0^* = 0$ and Θ^f is independent of its third argument when its second argument is zero. The second inequality follows because $(p_0^*, x', \Theta^f(x_0^*, B_1^f(x'), x'))$ is a feasible strategy when $z = 1$.

Finally, by Berge's Maximum Theorem, the vendor's optimal profit is continuous in α for both values of z. Therefore, $\pi(p_1^*, x_1^*, \Theta_1^*) - \pi(p_0^*, x_0^*, \Theta_0^*)$ is a continuous function of α. In other words, there exists an $\tilde{\alpha} < 1$ such that for $\alpha \in (\tilde{\alpha}, 1)$, $\pi(x_1^*, \Theta_1^*) - \pi(x_0^*, \Theta_0^*) > 0$. □

A.10 Proof of Proposition 15

When β is exogenous, $\bar{\theta}$ is simply defined as $\bar{\theta}(p, x, z) = \sqrt{\frac{p}{\kappa(\alpha,\beta,x,z)}}$; see (3). The only difference is that β is now a fixed quantity. Let the vendor's profit be denoted as: $\pi(p, x, z) = p\left(1 - \bar{\theta}(p, x, z)\right) + L\log(1 - x)$. Observe that the proof of Lemma 2 does not make use of the endogeneity of β and, therefore, applies in this setting as well. It follows that vendor's optimization problem has an optimal solution. Using the first order optimality conditions, the optimal p^* can be computed in a straightforward manner and it can be verified that $\bar{\theta}(p^*, x) = \frac{2}{3}$. Next, we compare the profits for the different patch restriction policies: *i.e.*, $z = 0$ versus $z = 1$. Now, it is easy to verify that κ is non-increasing in z. Therefore, for any (p, x), $\bar{\theta}(p, x, z)$ is non-decreasing in z. It follows that $\bar{\theta}(p, x, 1) \geq \bar{\theta}(p, x, 0)$. As a result, $\pi(p, x, 1) \leq \pi(p, x, 0)$. □

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